One-Shot Manipulation of Dynamical Quantum Resources

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We develop a unified framework to characterize one-shot transformations of dynamical quantum resources in terms of resource quantifiers, establishing universal conditions for exact and approximate transformations in general resource theories. Our framework encompasses all dynamical resources represented as quantum channels, including those with a specific structure—such as boxes, assemblages, and measurements—thus immediately applying in a vast range of physical settings. For the particularly important manipulation tasks of distillation and dilution, we show that our conditions become necessary and sufficient for broad classes of important theories, enabling an exact characterization of these tasks and establishing a precise connection between operational problems and resource monotones based on entropic divergences. We exemplify our results by considering explicit applications to quantum communication, where we obtain exact expressions for one-shot quantum capacity and simulation cost assisted by no-signaling, separability-preserving, and positive partial transpose–preserving codes; as well as to nonlocality, contextuality, and measurement incompatibility, where we present operational applications of a number of relevant resource measures.

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Introduction.—Manipulating different resources under physical restrictions underpins many quantum technologies, and the precise understanding of physically realizable transformations is a fundamental question both from theoretical and practical points of view. Quantum resource theories [1–3] provide a platform through which the quantification and manipulation of resources can be explicitly considered, enabling the study of resource manipulation in a range of settings of interest [4–7].

Because of the inherent generality of the framework of resource theories, the formalism could be expected to provide unifying results on resource manipulation that hold for diverse classes of resources. Developing such versatile *general resource theories* allows one to extract common features shared by different physical phenomena and clarify the peculiarities that differentiate one setting from others [7–25]. However, previous approaches to these problems suffer from a number of limitations. Many works have focused on the static, rather than dynamical resources, in the sense that only the manipulation of quantum states was considered, and a more general approach that incorporates the ability to manipulate the dynamics of the systems was not established. Recent works have begun to describe channel-based theories [14,22,23,26–30], but often focused on the investigation of specific theories such as entanglement, coherence, or quantum memories [31–43], or obtained results that only apply in the idealized asymptotic limit [26]. The characterization of dynamical resource theories is significantly more complex than the manipulation of underlying states, and many questions remain unanswered. Notably, the precise characterization of convertibility between two quantum channels in the practical *one-shot* scenario has been an outstanding problem to be addressed.

An important aspect of dynamical resources is that they can describe a much broader range of settings than the commonly considered manipulation of quantum states with general quantum channels. For instance, Bell nonlocality [44,45] as well as quantum contextuality [46,47] have been investigated within formal resource-theoretic settings [48–52], but the specific restrictions on the structure of channels allowed in these settings prevented them from being integrated into most of the previous general quantum resource frameworks. The recent works of Refs. [53,54] considered an approach to quantum nonlocality that encompasses any type of input and output systems, suggesting that the establishment of a broader framework of channel resource theories might be possible, and opening up the potential to address the manipulation of dynamical resources in a unified manner.

Here, we achieve such a unified description by providing a universal characterization of one-shot resource transformations with finite error in terms of fundamental resource measures, valid in general resource theories of quantum channels. By allowing arbitrary restrictions on the set of channels under consideration, our results apply not only to settings previously studied in resource theories of channels—e.g., entanglement, coherence, magic, quantum thermodynamics, and quantum communication—but also to other dynamical resources such as Bell nonlocality, contextuality, steering [55,56], measurement incompatibility [57], and many others [53,54], offering a very general quantitative description of the fundamental task of resource manipulation. As an important subclass of manipulation tasks, we study resource distillation and dilution, where we present additional bounds and results that provide necessary and sufficient conditions for resource conversion in many relevant cases. This provides an exact characterization of optimal one-shot rates achievable in these tasks, in which case important resource measures—such as resource robustness and hypothesis testing relative entropy—are endowed with explicit operational meaning. We further establish tight benchmarks on the achievable fidelity of distillation. Finally, we discuss insights provided by our general results into several physical scenarios. We first apply our results to quantum communication and find exact expressions for quantum capacity and simulation cost for communication assisted by no-signaling codes and codes preserving separability or the positivity of the partial transpose (PPT). We then discuss the application to nonlocality, contextuality, and measurement incompatibility, where we link resource measures previously introduced in other settings to approximate resource transformations.

We focus on discussing our main results below, and the technical proofs are deferred to the Supplemental Material [58].

Manipulation of dynamical resources.—Let $\mathbb{O}_{\rm all}$ be the set of valid channels allowed in the given physical setting; this can be the set of all quantum channels or a subset thereof, allowing us to take into consideration possible restrictions on the types of channels being manipulated. Each resource theory also designates a subset of channels that are considered to be available for free, and we denote the given free channels as $\mathbb{O} \subseteq \mathbb{O}_{\rm all}$. We impose mild assumptions that the underlying Hilbert spaces are always finite dimensional and, for a fixed dimension, the set of free channels $\mathbb{O}$ is convex and closed [7].

General transformations of quantum channels are described by quantum superchannels [65]. Since superchannels need not preserve specific channel structures in general [53], we consider the subset of superchannels that map the set of allowed channels $\mathbb{O}_{\rm all}$ to allowed output channels $\mathbb{O}'_{\rm all}$, defined as $\mathbb{S}_{\rm all} \coloneqq \{\Theta\colon \mathbb{O}_{\rm all} \to \mathbb{O}'_{\rm all}\}$. We can now take a subset of $\mathbb{O}'_{\rm all}$ and consider it as the free channels in the output space, which we denote by $\mathbb{O}'$. A subset of $\mathbb{S}_{\rm all}$ serves as the set of free transformation that can be used for the manipulation of resources. The standard requirement for any free operation is that it should not generate any resource, i.e., it should not create any costly channel out of a free one. We consider the maximal set satisfying this condition, $\mathbb{S} \coloneqq \{\Theta \in \mathbb{S}_{\rm all} | \Theta(\mathcal{M}) \in \mathbb{O}' \ \forall \ \mathcal{M} \in \mathbb{O}\}$.

Our goal is to find the conditions for the transformation from $\mathcal{E}$ to $\mathcal{N}$ using free superchannels in $\mathbb{S}$, given any two channels $\mathcal{E} \in \mathbb{O}_{\rm all}$ and $\mathcal{N} \in \mathbb{O}'_{\rm all}$. In practice, the transformation can often only be achieved approximately, especially in nonasymptotic resource manipulation. To evaluate the inevitable error, we consider the worst-case fidelity [66,67] defined for two channels $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{O}_{\rm all}$ as $F(\mathcal{E}_1, \mathcal{E}_2) \coloneqq \min_\rho F(\mathrm{id} \otimes \mathcal{E}_1(\rho), \mathrm{id} \otimes \mathcal{E}_2(\rho))$ where $F(\rho, \sigma) \coloneqq \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$ is the fidelity.

Our aim will be to characterize the conditions on resource transformation through the resource contents of the given channels. To this end, we introduce two types of resource measures. They both belong to, or are closely related to, the class of one-shot entropic quantities [68] and, in particular, channel divergences [69,70]. The first class is known as the robustness measures [71] defined for any $\mathcal{E} \in \mathbb{O}_{\rm all}$ as

$$R_{\mathbb{O};\tilde{\mathbb{O}}}(\mathcal{E}) \coloneqq \inf\left\{1 + r \,\middle|\, \frac{\mathcal{E} + r\mathcal{M}}{1 + r} \in \mathbb{O}, \mathcal{M} \in \tilde{\mathbb{O}}\right\}, \quad (1)$$

where $\tilde{\mathbb{O}} \subseteq \mathbb{O}_{\rm all}$ is some set of channels containing $\mathbb{O}$. The extreme case $\tilde{\mathbb{O}} = \mathbb{O}_{\rm all}$ is known as *generalized robustness* [14,72,73] and corresponds to the max-relative entropy [74], hence we denote it by $R_{\max,\mathbb{O}} \coloneqq R_{\mathbb{O};\mathbb{O}_{\rm all}}$. The other case of interest is the *standard robustness* $R_{s,\mathbb{O}} \coloneqq R_{\mathbb{O};\mathbb{O}}$ [30,41]. We also define the smooth robustness $R^\epsilon_{\mathbb{O};\tilde{\mathbb{O}}}(\mathcal{E}) \coloneqq \min\{R_{\mathbb{O};\tilde{\mathbb{O}}}(\mathcal{E}') | F(\mathcal{E}', \mathcal{E}) \geq 1 - \epsilon, \mathcal{E}' \in \mathbb{O}_{\rm all}\}$ for $0 \leq \epsilon \leq 1$. The other type of measure, based on the hypothesis testing relative entropy [69,70,75,76], is defined for $\mathcal{E} \in \mathbb{O}_{\rm all}$ as

$$R^\epsilon_{H,\mathbb{O}}(\mathcal{E}) \coloneqq \min_{\mathcal{M} \in \mathbb{O}} \max_{\psi} R^\epsilon_H(\mathrm{id} \otimes \mathcal{E}(\psi) || \mathrm{id} \otimes \mathcal{M}(\psi)), \quad (2)$$

where $R^\epsilon_H(\rho||\sigma) \coloneqq \max_{\substack{0 \leq P \leq 1 \\ \mathrm{Tr}(P\rho) \geq 1-\epsilon}} \mathrm{Tr}(P\sigma)^{-1}$ and the optimization is restricted to pure input states ψ without loss of generality. This entropic quantity characterizes the distinguishability between $\mathcal{E}$ and the channels in the set $\mathbb{O}$. The case of $\epsilon = 0$ is known as the min-relative entropy [74], denoted as $R_{\min,\mathbb{O}}(\mathcal{E})$.

It is also useful to introduce two classifications for the given theory depending on the properties of $\mathbb{O}$. We say that $\mathbb{O}$ is *full dimensional* if $\mathrm{span}(\mathbb{O})$ contains all channels in $\mathbb{O}_{\rm all}$, and *reduced dimensional* otherwise [18]. Intuitively, in full-dimensional theories, the set $\mathbb{O}$ is of full measure, meaning that $R_{s,\mathbb{O}}(\mathcal{E}) < \infty$ for all $\mathcal{E}$; examples include the theory of entanglement or local operations and shared randomness. On the other hand, reduced-dimensional theories are equipped with a set of free channels $\mathbb{O}$ of zero measure, and the standard robustness $R_{s,\mathbb{O}}$ can diverge, examples of which include the theory of coherence, asymmetry, and quantum thermodynamics. In order to characterize such resources, we will often need to consider an optimization with respect to $\mathrm{aff}(\mathbb{O})$, the affine hull of $\mathbb{O}$ [18,77,78], and we define, in particular, $R^{\epsilon}_{H,\mathrm{aff}(\mathbb{O})}$ (respectively, $R_{\min,\mathrm{aff}(\mathbb{O})}$) as the hypothesis testing entropy (min-relative entropy) minimized over $\mathrm{aff}(\mathbb{O})$ instead of $\mathbb{O}$.

We can now state our main results that connect the resource monotones with general one-shot resource transformations.

Theorem 1.—Let $\mathcal{E} \in \mathbb{O}_{\mathrm{all}}$ and $\mathcal{N} \in \mathbb{O}'_{\mathrm{all}}$. If there exists a free superchannel $\Theta \in \mathbb{S}$ such that $F(\Theta(\mathcal{E}), \mathcal{N}) \geq 1-\epsilon$, then for any monotone $\mathfrak{R}_{\mathbb{O}}$ it holds that $\mathfrak{R}_{\mathbb{O}}(\mathcal{E}) \geq \mathfrak{R}^{\epsilon}_{\mathbb{O}'}(\mathcal{N})$, as well as that $\mathfrak{R}^{\delta}_{\mathbb{O}}(\mathcal{E}) \geq \mathfrak{R}^{2(\sqrt{\delta}+\sqrt{\epsilon})}_{\mathbb{O}'}(\mathcal{N})$ for any $0 \leq \delta \leq 1$ where $\mathfrak{R}^{\epsilon}_{\mathbb{O}}(\mathcal{E}) \coloneqq \min\{\mathfrak{R}_{\mathbb{O}}(\mathcal{E}') | F(\mathcal{E}', \mathcal{E}) \geq 1-\epsilon, \mathcal{E}' \in \mathbb{O}_{\mathrm{all}}\}$.

Conversely, for any choices of ϵ, $\delta \geq 0$ such that $\epsilon + 2\delta < 1$, there exists a free superchannel $\Theta \in \mathbb{S}$ such that $F(\Theta(\mathcal{E}), \mathcal{N}) \geq 1-\epsilon-2\delta$ if $R^{\delta}_{H,\mathbb{O}}(\mathcal{E}) \geq R^{\epsilon}_{s,\mathbb{O}'}(\mathcal{N})$ or if $R^{\delta}_{H,\mathrm{aff}(\mathbb{O})}(\mathcal{E}) \geq R^{\epsilon}_{\max,\mathbb{O}'}(\mathcal{N})$.

Here, the parameters ϵ, δ can be used to study the tradeoffs between the error allowed in the transformation and the values of the smoothed monotones $\mathfrak{R}^{\epsilon}_{\mathbb{O}}$ of both the input and the output channel.

Notice that we provided two alternative achievability conditions: one using the hypothesis testing measure $R_{H,\mathbb{O}}$ and the standard robustness $R_{s,\mathbb{O}'}$, and one using the affine hypothesis testing measure $R_{H,\mathrm{aff}(\mathbb{O})}$ and the robustness $R_{\max,\mathbb{O}'}$. The reason for this is that the former condition will typically trivialize in reduced-dimensional theories, while the latter condition trivializes for full-dimensional theories.

Theorem 1 establishes the conditions for general resource transformation universally applicable to any resource theory, including ones with specific structures of allowed channels, reflected by choosing appropriate sets $\mathbb{O}_{\mathrm{all}}$ and $\mathbb{O}'_{\mathrm{all}}$. In the special case of manipulating quantum states (channels with trivial input), we recover the results of Ref. [20] and extend them to reduced-dimensional theories. We stress that the monotones are all convex optimization problems and they reduce to computable semidefinite programs when $\mathbb{O}$ is characterized by semidefinite constraints [58].

Distillation and dilution.—Two of the most important classes of resource transformation tasks are resource distillation, where general resources are transformed into some reference target resources, and dilution, where the reference target resources are used to synthesize a given channel through free transformations. In particular, one is often interested in two quantities: the *distillable resource* $d^{\epsilon}_{\mathbb{O}}(\mathcal{E})$ and the *resource cost* $c^{\epsilon}_{\mathbb{O}}(\mathcal{E})$; choosing a suitable class of target reference channels $\mathbb{T} \subseteq \mathbb{O}'_{\mathrm{all}}$, we can define

$$\begin{aligned} d^{\epsilon}_{\mathbb{O}}(\mathcal{E}) &\coloneqq \sup\{\mathfrak{R}_{\mathbb{O}'}(\mathcal{T}) | F(\Theta(\mathcal{E}), \mathcal{T}) \geq 1-\epsilon, \mathcal{T} \in \mathbb{T}, \Theta \in \mathbb{S}\}, \\ c^{\epsilon}_{\mathbb{O}}(\mathcal{E}) &\coloneqq \inf\{\mathfrak{R}_{\mathbb{O}'}(\mathcal{T}) | F(\mathcal{E}, \Theta(\mathcal{T})) \geq 1-\epsilon, \mathcal{T} \in \mathbb{T}, \Theta \in \mathbb{S}\}, \end{aligned} \tag{3}$$

where $\mathfrak{R}_{\mathbb{O}'}$ refers to any chosen monotone—for example, $R_{\min,\mathbb{O}'}$, $R_{\max,\mathbb{O}'}$, or $R_{s,\mathbb{O}'}$. In the discussion below, we will fix $\mathfrak{R}_{\mathbb{O}'} = R_{\min,\mathbb{O}'}$ for simplicity. The target channels $\mathbb{T}$ are often chosen as multiple copies of some fixed reference channel, but we allow for broader types of targets.

Notably, under suitable conditions on the reference channels $\mathbb{T}$, the necessary and sufficient conditions of Theorem 1 coincide, yielding a precise characterization of the resource cost.

Corollary 2.—If the chosen reference set satisfies $R_{\min,\mathbb{O}'}(\mathcal{T}) = R_{s,\mathbb{O}'}(\mathcal{T})\ \forall\, \mathcal{T} \in \mathbb{T}$, then it holds that $c^{\epsilon}_{\mathbb{O}}(\mathcal{E}) = \lceil R^{\epsilon}_{s,\mathbb{O}}(\mathcal{E}) \rceil_{\mathbb{T}}$.

Similarly, if the chosen reference set obeys $R_{\min,\mathrm{aff}(\mathbb{O}')}(\mathcal{T}) = R_{\max,\mathbb{O}'}(\mathcal{T})\ \forall\, \mathcal{T} \in \mathbb{T}$, then it holds that $c^{\epsilon}_{\mathbb{O}}(\mathcal{E}) = \lceil R^{\epsilon}_{\max,\mathbb{O}}(\mathcal{E}) \rceil_{\mathbb{T}}$.

Here, we used the notation $\lceil \cdot \rceil_{\mathbb{T}}$ to indicate the smallest value greater than or equal to the argument for which there exists a corresponding channel $\mathcal{T} \in \mathbb{T}$—this is required, for instance, when $\mathbb{T}$ forms a discrete set (see Ref. [58]).

The result establishes an operational meaning of the measures $R^{\epsilon}_{s,\mathbb{O}}$ or $R^{\epsilon}_{\max,\mathbb{O}}$ as long as the conditions are satisfied. This raises the question: when does a choice of reference channels $\mathbb{T}$ satisfying $R_{\min,\mathbb{O}}(\mathcal{T}) = R_{s,\mathbb{O}}(\mathcal{T})$ or $R_{\min,\mathrm{aff}(\mathbb{O})}(\mathcal{T}) = R_{\max,\mathbb{O}}(\mathcal{T})$ exist? It emerges that this is a commonly occurring feature of resource theories. For instance, it is satisfied by relevant choices of target channels in resource theories of quantum memories and communication, immediately providing a characterization of one-shot simulation cost of channels. When discussing the transformations of quantum states, it was shown that in *any* convex resource theory there exists maximal "golden states" ϕ_{gold} such that $R_{\min,\mathbb{O}}(\phi_{\mathrm{gold}}) = R_{\max,\mathbb{O}}(\phi_{\mathrm{gold}})$ [18], and indeed such states satisfy all requirements of the Corollary in theories such as quantum entanglement or quantum coherence. In the case of quantum state manipulation, our result recovers the considerations of Ref. [16], where a general framework for quantum state resources was established.

We now turn to the case of distillation. Here, we can improve the bound in Theorem 1 and obtain an alternative necessary condition. Importantly, distillation is often understood as the *purification* of noisy resources, in which case it is natural to consider pure reference channels $\mathbb{T}$—for instance, if the input space and output spaces coincide, unitary channels can serve as targets, while if the input space is trivial, then pure-state preparation channels can be regarded as targets. The property needed for $\mathcal{N}$ to serve as the reference resource is that the output states for pure input states remain pure. In such cases, we obtain the following general conditions.

Theorem 3.—Let $\mathcal{E} \in \mathbb{O}_{\mathrm{all}}$, $\mathcal{N} \in \mathbb{O}'_{\mathrm{all}}$ and suppose $\mathrm{id} \otimes \mathcal{N}(\psi)$ is pure for any pure state ψ. If there exists a free superchannel $\Theta \in \mathbb{S}$ such that $F(\Theta(\mathcal{E}), \mathcal{N}) \geq 1-\epsilon$, then it holds that $R^{\epsilon}_{H,\mathbb{O}}(\mathcal{E}) \geq R_{\min,\mathbb{O}'}(\mathcal{N})$ and $R^{\epsilon}_{H,\mathrm{aff}(\mathbb{O})}(\mathcal{E}) \geq R_{\min,\mathrm{aff}(\mathbb{O}')}(\mathcal{N})$.

Conversely, there exists a free superchannel $\Theta \in \mathbb{S}$ such that $F(\Theta(\mathcal{E}), \mathcal{N}) \geq 1 - \epsilon - \delta$ if $R^{\epsilon}_{H,\mathbb{O}}(\mathcal{E}) \geq R^{\delta}_{s,\mathbb{O}'}(\mathcal{N})$ or if $R^{\epsilon}_{H,\mathrm{aff}(\mathbb{O})}(\mathcal{E}) \geq R^{\delta}_{\max,\mathbb{O}'}(\mathcal{N})$.

Theorem 3 adds a useful alternative characterization for distillation to the general condition provided by Theorem 1. In particular, similarly to the case of dilution, we can obtain the following.

Corollary 4.—Consider any reference set $\mathbb{T}$ such that $\mathrm{id} \otimes \mathcal{T}(\psi)$ is pure for any pure ψ and any $\mathcal{T} \in \mathbb{T}$.

If $\mathbb{T}$ also satisfies $R_{\min,\mathbb{O}}(\mathcal{T}) = R_{s,\mathbb{O}}(\mathcal{T})\ \forall\ \mathcal{T} \in \mathbb{T}$, then it holds that $d^{\epsilon}_{\mathbb{O}}(\mathcal{E}) = \lfloor R^{\epsilon}_{H,\mathbb{O}}(\mathcal{E}) \rfloor_{\mathbb{T}}$.

Similarly, if the chosen reference set obeys $R_{\min,\mathrm{aff}(\mathbb{O})}(\mathcal{T}) = R_{\max,\mathbb{O}}(\mathcal{T})\ \forall\, \mathcal{T} \in \mathbb{T}$, then it holds that $d^{\epsilon}_{\mathbb{O}}(\mathcal{E}) = \lfloor R^{\epsilon}_{H,\mathrm{aff}(\mathbb{O})}(\mathcal{E}) \rfloor_{\mathbb{T}}$.

This establishes a precise characterization of distillable resource in any resource theory for suitable target channels $\mathbb{T}$, and furthermore gives an exact operational meaning to the resource measures $R^{\epsilon}_{H,\mathbb{O}}$ and $R^{\epsilon}_{H,\mathrm{aff}(\mathbb{O})}$ in the task of distillation. When the manipulated objects are quantum states, we recover the results of Refs. [16,18].

In some cases, distillation with the desired precision might not be possible. It is then of interest to instead ask how close one can approximate the chosen target channel, that is, characterize the maximal achievable *fidelity of distillation*. We can adapt our methods to obtain close upper and lower bounds for this quantity. Importantly, the bounds become tight for relevant reference states obeying conditions as in Corollary 4, allowing us to provide an exact expression for the fidelity of distillation. We discuss the full details in Ref. [58].

Another bound on distillation fidelity was recently presented with respect to the so-called resource weight [22,23]. Our approach allows one to extend the insight from these works to theories with arbitrary channel structures $\mathbb{O}_{\mathrm{all}}$, enabling an operational application of the corresponding weight measures, some of which were previously introduced in other contexts [49,79–83].

One can obtain additional results in the characterization of distillation and dilution in special cases of resource theories, for example, when the given theory is concerned with an underlying state-based resource. We discuss such cases in Ref. [58].

Quantum communication.—A central problem in quantum communication is to manipulate a given channel to enhance its communication capabilities using resources available to both parties. Quantum capacity [84–86] and simulation cost [86–88] are important figures of merit to evaluate the operational capability of quantum channels, and their one-shot characterization received considerable attention recently [29,89–93]. These tasks are precisely channel distillation and dilution where the reference resource is the identity channel, and the sets of free channels and free superchannels specify the accessible resources for the sender and receiver. Formally, we define the one-shot quantum capacity and simulation cost with the set of free superchannels $\mathbb{S}$ as

$$Q^{\epsilon}_{\mathbb{S}}(\mathcal{E}) \coloneqq \max\{\log d \,|\, \exists \Theta \in \mathbb{S}, F(\Theta(\mathcal{E}), \mathrm{id}_d) \geq 1 - \epsilon\},$$
$$C^{\epsilon}_{\mathbb{S}}(\mathcal{E}) \coloneqq \min\{\log d \,|\, \exists \Theta \in \mathbb{S}, F(\Theta(\mathrm{id}_d), \mathcal{E}) \geq 1 - \epsilon\}. \quad (4)$$

We can then use our results to immediately obtain an exact characterization of these quantities in relevant settings. For example, when $\mathbb{O}$ is the set of separable channels $\mathbb{O}_{\mathrm{SEP}}$ whose Choi states are separable, this setting corresponds to communication with separability-preserving codes $\mathbb{S} = \mathbb{S}_{\mathrm{SEP}}$. This is a full-dimensional theory and it holds that $R_{\min,\mathbb{O}_{\mathrm{SEP}}}(\mathrm{id}_d) = R_{s,\mathbb{O}_{\mathrm{SEP}}}(\mathrm{id}_d) = d$ [41]. Then, Corollaries 2 and 4 provide a complete characterization of one-shot quantum capacity and simulation cost as $Q^{\epsilon}_{\mathbb{S}_{\mathrm{SEP}}}(\mathcal{E}) = \log\lfloor R^{\epsilon}_{H,\mathbb{O}_{\mathrm{SEP}}}(\mathcal{E}) \rfloor$ and $C^{\epsilon}_{\mathbb{S}_{\mathrm{SEP}}}(\mathcal{E}) = \log\lceil R^{\epsilon}_{s,\mathbb{O}_{\mathrm{SEP}}}(\mathcal{E}) \rceil$, the latter of which recovers a result of Ref. [41]. Similar results apply to the setting of communication assisted by codes preserving the positivity of the partial transpose (PPT), where $\mathbb{O}_{\mathrm{PPT}}$ is the set of PPT channels [89,94]. We analogously obtain $Q^{\epsilon}_{\mathbb{S}_{\mathrm{PPT}}}(\mathcal{E}) = \log\lfloor R^{\epsilon}_{H,\mathbb{O}_{\mathrm{PPT}}}(\mathcal{E}) \rfloor$ and $C^{\epsilon}_{\mathbb{S}_{\mathrm{PPT}}}(\mathcal{E}) = \log\lceil R^{\epsilon}_{s,\mathbb{O}_{\mathrm{PPT}}}(\mathcal{E}) \rceil$. Interestingly, $R^{\epsilon}_{H,\mathbb{O}_{\mathrm{PPT}}}$ appeared as a bound in Ref. [90].

Another example is the case when $\mathbb{O}$ is the set of replacement channels $\mathbb{O}_R \coloneqq \{\mathcal{R}_\sigma | \mathcal{R}_\sigma(\cdot) = \mathrm{Tr}(\cdot)\sigma\}$, where $\mathbb{S}$ becomes the set of channel transformations assisted by no-signaling (NS) correlations, $\mathbb{S}_{\mathrm{NS}}$ [29,95]. Since $\mathbb{O}_R$ is closed under linear combinations, it is reduced dimensional. We also have $R_{\min,\mathrm{aff}(\mathbb{O}_R)}(\mathrm{id}_d) = R_{\max,\mathbb{O}_R}(\mathrm{id}_d) = d^2$, and our results immediately give $Q^{\epsilon}_{\mathbb{S}_{\mathrm{NS}}}(\mathcal{E}) = \frac{1}{2}\log\lfloor R^{\epsilon}_{H,\mathrm{aff}(\mathbb{O}_R)}(\mathcal{E}) \rfloor$ and $C^{\epsilon}_{\mathbb{S}_{\mathrm{NS}}}(\mathcal{E}) = \frac{1}{2}\log\lceil R^{\epsilon}_{\max,\mathbb{O}_R}(\mathcal{E}) \rceil$. The one-shot NS-assisted quantum capacity was obtained in Ref. [91] in the form of a semidefinite program; our result identifies it with the affine hypothesis testing relative entropy, providing an operational meaning to this resource measure. The one-shot NS simulation cost was obtained in Refs. [29,92], which is recovered by our general approach as a special case. Furthermore, we can quantify exactly the fidelity of NS-assisted coding [58], recovering a result of Ref. [89].

Our methods apply also to the study of the entanglement of bipartite channels [34,96], where the target resources are maximally entangled states in the underlying state-based resource theory. We then establish exact one-shot rates of channel manipulation under PPT- and separability-preserving superchannels (see Ref. [58]).

Nonlocality and contextuality.—Quantum nonlocality has been a major subject of study not only as a key feature of quantum theory, but also as a useful resource in a number of operational tasks [97–102]. The latter view motivates a precise understanding of the manipulation of nonlocal resources [49,55,103,104], but the characterization of such one-shot transformations has remained elusive. Our framework encompasses this scenario by choosing $\mathbb{O}_{\rm all}$ to be the set of no-signaling channels, where classical input or output systems are represented through dephasing in a given basis, and taking $\mathbb{O}$ to be the channels that can be constructed by local operations and shared randomness. This includes not only the standard setting of Bell nonlocality (where such channels are the classical "boxes"), but also more general resources such as steering (where channels represent "assemblages") [53,54]. Our results then provide an operational application of nonlocality measures individually introduced in different settings of nonlocality [49,105] and unified in Refs. [53,54], which we relate to one-shot resource transformations (see also Ref. [50]). We also introduce new monotones to this setting, which can add further insights into feasible manipulation of nonlocal resources. For instance, since a noisy box of the form $\mathcal{B} = (1-\epsilon)\mathcal{B}_{\rm PR} + \epsilon\mathcal{B}_{\rm free}$, where $\mathcal{B}_{\rm PR}$ is the Popescu-Rohrlich (PR) box [106] and $\mathcal{B}_{\rm free} \in \mathbb{O}$ is some local box, has $R_{\min,\mathbb{O}}(\mathcal{B}^{\otimes n}) = 1\ \forall\ n$, we recover the fact that it cannot be distilled to other fundamental resources such as the PR box even when multiple copies of the box $\mathcal{B}$ are available [103,107]. In Ref. [58], we numerically evaluate the resource measures for a special class known as isotropic boxes [108] where we explicitly observe this property.

Notably, our framework can also be applied to another related phenomenon known as quantum contextuality, which also serves as an operational resource [109–115]. Namely, we consider the set of all classical-classical channels for consistent boxes [81] as $\mathbb{O}_{\rm all}$ and the set of channels corresponding to noncontextual boxes as $\mathbb{O}$. Our results characterize the exact and approximate box transformations with operations that do not create contextuality, offering a new perspective to the recent resource-theoretic framework [51,52,116], as well as providing operational application of the robustness of contextuality [117–119], contextual fraction [81,82,120], and hypothesis testing measures introduced in this work.

Measurement incompatibility.—Measurement incompatibility refers to the impossibility of simultaneous measurement and is closely related to the aforementioned phenomena such as Bell nonlocality and steering [83,121,122]. The set of POVMs $\{M_{a|x}\}$, where $M_{a|x}$ is the POVM element with outcome a for the measurement labeled by setting x, is called compatible (or jointly measurable) if there exists a parent measurement $\{P_i\}$ and a conditional probability distribution $\{q(a|x,i)\}$ such that $M_{a|x} = \sum_i q(a|x,i)P_i$.

Our formalism can handle scenarios where resources take the form of ensembles by incorporating the classical labels of the ensembles into the description of $\mathbb{O}_{\rm all}$. In the case of measurement incompatibility, $\mathbb{O}_{\rm all}$ represents the set of channels corresponding to POVMs, while $\mathbb{O} \subseteq \mathbb{O}_{\rm all}$ represents the set of compatible POVMs (see Ref. [58]). This form allows one to apply our results to this setting and characterize the approximate one-shot transformation of incompatible sets of measurements, complementing the previous works which focused on exact transformation with smaller sets of free operations in different approaches [123,124] and providing operational applications of the related measures [83,125–127]. Although the robustness and weight measures are usually defined at the level of POVMs, we show in Ref. [58] that they coincide with the channel-based measures defined in our framework, allowing one to carry over the previous analyses to characterize resource transformations. We also note that the discussion here can be straightforwardly extended to channel incompatibility [57,128], which includes measurement incompatibility as a special case.

Conclusions.—We established fundamental bounds on the transformations between general dynamical quantum resources in the one-shot regime. We tightly characterized the ability to manipulate resources by providing conditions for convertibility in terms of the robustness and hypothesis testing measures. In particular, under suitable assumptions, we established an exact quantification of the one-shot distillable resource and one-shot resource cost of general channels, giving a precise operational interpretation to the considered monotones in these important tasks. This not only extends and unifies previous specialized results [34,36,41,89,91,96], but also sheds light on the general structure of dynamical resource theories by providing a common description of their operational aspects. Besides contributing to the theory of quantum resources, our methods find direct practical use, as we exemplified with several explicit applications.

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Note added.—During the completion of this manuscript, we became aware of two related works, Ref. [129] by Kim *et al.* and Ref. [130] by Yuan *et al.*, where the authors independently obtained results overlapping with some of our findings. The former work considers one-shot distillation and dilution of quantum channel entanglement, which coincides with our characterization of these tasks in Corollaries 2 and 4 (see Ref. [58]), while the latter work introduces a general framework for one-shot channel distillation and dilution which again corresponds to our Corollaries 2 and 4 for the cases of quantum channel resources.

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