Overcoming entropic limitations on asymptotic state transformations through probabilistic protocols

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The quantum relative entropy is known to play a key role in determining the asymptotic convertibility of quantum states in general resource-theoretic settings, often constituting the unique monotone that is relevant in the asymptotic regime. We show that this is no longer the case when one allows stochastic protocols that may only succeed with some probability, in which case the quantum relative entropy is insufficient to characterize the rates of asymptotic state transformations, and a new entropic quantity based on a regularization of the Hilbert projective metric comes into play. Such a scenario is motivated by a setting where the cost associated with transformations of quantum states, typically taken to be the number of copies of a given state, is instead identified with the size of the quantum memory needed to realize the protocol. Our approach allows for constructing transformation protocols that achieve strictly higher rates than those imposed by the relative entropy. Focusing on the task of resource distillation, we give broadly applicable strong converse bounds on the asymptotic rates of probabilistic distillation protocols, and show them to be tight in relevant settings such as entanglement distillation with nonentangling operations. This generalizes and extends previously known limitations that are only applicable to deterministic protocols. Our methods are based on recent results for probabilistic one-shot transformations as well as a new asymptotic equipartition property for the projective relative entropy.

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I. INTRODUCTION

Transformations of quantum states underlie the majority of quantum information processing protocols, and understanding their capabilities and limitations is one of the fundamental problems posed in quantum information science. The ultimate form of such transformations is often understood to be the limit of infinitely many independent and identically distributed copies of a given quantum state being coherently manipulated. Although a somewhat idealized scenario, such limits often enjoy simplified properties and have found a multitude of uses in the characterization of different quantum phenomena [1,2]. One of the appeals of these approaches is that their asymptotic transformation rates are naturally described by various entropic quantities, giving an explicit operational meaning to measures such as the quantum relative entropy [3–7] or the regularized relative entropy of entanglement [8,9]. There are, however, many assumptions hidden within these standard Shannon-theoretic approaches.

In computing the asymptotic rates of quantum state manipulation, only one quantity is relevant: how many copies of a given state ρ need to be produced per copy of a desired state ω in order to realize the transformation $\rho \to \omega$. This is conceptually appealing, but arguably not fully indicative of practical restrictions on state manipulation: this approach assumes that we can coherently manipulate any number of copies $\rho^{\otimes n}$, and indeed there is no cost associated with the size of the quantum memory needed to perform such a manipulation. Ideally, the "cost function" associated with a transformation should take more parameters into account, reflecting also the difficulty in manipulating many quantum states simultaneously to realize multicopy operations.

Here we propose a framework motivated by the opposite point of view: instead of taking into account the number of copies of ρ needed for the transformation, let us focus purely on the quantum memory cost—that is, how many copies of ρ need to be manipulated at once. The biggest difference between this setting and conventional approaches is that now we want to avoid manipulating too many states at once, but it does not matter *how many times* we do it. Practically, this setting becomes relevant in a situation where generating copies of ρ is much less expensive than storing and processing them. This is in many respects the case today, as the best sources can generate more than $\approx 10^5$ entangled photon pairs

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per second per mW of power invested [10,11], while the best quantum processors cannot store and process more than a few tens of qubits. From the conceptual standpoint, our paradigm can be compared with algorithmic complexity classes such as PSPACE, which ignore the time needed to evaluate a program, but tightly constrain its space complexity.

This approach allows us to employ repeat-until-success probabilistic transformation protocols—potentially more powerful than typically employed deterministic ones—without incurring an additional cost. The limits of the power of such protocols in the asymptotic setting have not been explored yet, which is what this paper aims to address.

Results

We develop a technical toolset allowing for an exact characterization of the asymptotic limitations of probabilistic transformations of quantum states. We first introduce general converse bounds that constrain the performance of all probabilistic transformation schemes, valid in general resource-theoretic settings and thus allowing for broad applicability. Our bounds can be understood as the ultimate limitations of probabilistic transformations—no matter how successful one is in the effort to stochastically increase the performance of state manipulation protocols, the restrictions revealed here cannot be overcome. We show that the bounds can be tight in many relevant cases, and in particular in characterizing the distillation of resources such as quantum entanglement, as well as in all state transformations within the class of affine resource theories (including thermodynamics, coherence, and asymmetry). Notably, we exhibit probabilistic protocols that achieve rates strictly larger than bounds based on the quantum relative entropy: this shows that standard Shannon-theoretic approaches are insufficient to characterize the limitations of probabilistic transformations, as the latter can achieve performance forbidden by conventional entropic restrictions. An explicit example of such a behavior is provided by evaluating our bounds exactly for all isotropic states in entanglement theory.

On the technical side, our methods rely on the regularization of a quantum divergence based on the recently introduced projective robustness [12], and one of our main contributions is to develop an asymptotic equipartition property for this quantity.

We begin with a discussion of the setting in Sec. II. Our main results are divided into two parts: general upper bounds on the performance of probabilistic protocols, discussed in Sec. III, and the achievability results showing the tightness of those bounds, considered in Sec. IV. Explicit examples are then discussed in Sec. V. For simplicity, the detailed technical proofs are deferred to the Appendices.

II. PRELIMINARIES

A. Quantum resources

Adopting the framework of quantum resource theories [13], our task is to transform a given quantum state ρ into another state ω, using some restricted set of free operations $\mathcal{O}$ allowed within the given physical setting. An important property of any such free operations is that the set of states that can be prepared with the given operations—the free states $\mathcal{F}$—remains invariant under them, i.e., $\sigma \in \mathcal{F} \Rightarrow \Lambda(\sigma) \in \mathcal{F}$ for any channel $\Lambda \in \mathcal{O}$. We will take this as the only requirement that the free operations need to satisfy, which is often referred to as resource-nongenerating operations. This will ensure that the limitations obtained in our paper apply to other choices of physically relevant free operations, which are generally subsets of the resource-nongenerating ones.

Above, we have implicitly assumed that both the input and output spaces of the map Λ have associated sets of free states $\mathcal{F} \subseteq \mathcal{D}(\mathcal{H})$, with $\mathcal{D}(\mathcal{H})$ denoting all density operators acting on the given Hilbert space $\mathcal{H}$. More generally, when discussing transformations between many copies of quantum states, we will assume that each set $\mathcal{D}(\mathcal{H}^{\otimes n})$ has its own associated set of free states $\mathcal{F}_n$; we use $\mathcal{F}$ to refer to the whole family $(\mathcal{F}_n)_n$ for simplicity. Throughout this paper we assume that the underlying Hilbert space has finite dimension.

In order to ensure that the asymptotic quantities encountered throughout the paper are well defined, we follow [9] in introducing a set of basic axioms that the given resource theory should satisfy. These assumptions are obeyed in virtually every theory of interest, and for simplicity we assume that all sets $\mathcal{F}_n$ considered here satisfy them.

Axiom I. Each $\mathcal{F}_n$ is convex and closed.

Axiom II. There exists a full-rank state σ such that $\sigma^{\otimes n} \in \mathcal{F}_n$ for all n.

Axiom III. The sets $\mathcal{F}_n$ are closed under partial trace: if $\sigma \in \mathcal{F}_{n+1}$, then $\mathrm{Tr}_k\sigma \in \mathcal{F}_n$ for every $k \in \{1, \ldots, n+1\}$.

Axiom IV. The sets $\mathcal{F}_n$ are closed under tensor product: if $\sigma \in \mathcal{F}_n$ and $\sigma' \in \mathcal{F}_m$, then $\sigma \otimes \sigma' \in \mathcal{F}_{n+m}$.

B. Asymptotic transformation rates

A deterministic transformation rate is given by (see Fig. 1)

$$r(\rho \to \omega) := \sup\left\{r \,\middle|\, \lim_{n\to\infty} \inf_{\Lambda_n \in \mathcal{O}\cap\mathrm{CPTP}} \|\Lambda_n(\rho^{\otimes n}) - \omega^{\otimes\lfloor rn\rfloor}\|_1 = 0\right\}, \tag{1}$$

where we have emphasized that the allowed free operations belong to the set of completely positive and trace-preserving (CPTP) maps. Recall, however, that our setting allows us to employ probabilistic operations—which are certainly completely positive, but are only required to be trace nonincreasing [14,15]. We will refer to a probabilistic map Λ as free (resource nongenerating) if it satisfies

$$\sigma \in \mathcal{F} \quad \Rightarrow \quad \frac{\Lambda(\sigma)}{\mathrm{Tr}\,\Lambda(\sigma)} \in \mathcal{F}. \tag{2}$$

Let us then propose an alternative definition of an asymptotic transformation rate as

$$r_{\mathrm{prob}}(\rho \to \omega) := \sup\left\{r \,\middle|\, \lim_{n\to\infty} \inf_{\Lambda_n \in \mathcal{O}} \left\|\frac{\Lambda_n(\rho^{\otimes n})}{\mathrm{Tr}\,\Lambda_n(\rho^{\otimes n})} - \omega^{\otimes\lfloor rn\rfloor}\right\|_1 = 0\right\}. \tag{3}$$

Just as the conventional rate r, the probabilistic rate r_{prob} is defined in the limit $n \to \infty$, but it is ultimately concerned with how many copies of ω we can obtain per each single copy of ρ that we manipulate.

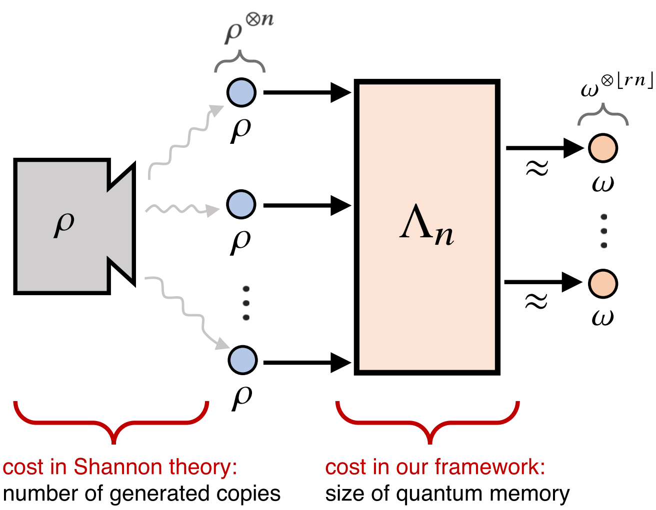

FIG. 1. Asymptotic state transformations. A general scheme for the conversion of a state ρ into another state ω takes n copies of ρ and manipulates them with some protocol Λ_n such that $\Lambda_n(\rho^{\otimes n}) \approx \omega^{\otimes\lfloor rn\rfloor}$, with the conversion becoming exact in the limit $n \to \infty$. In conventional quantum Shannon theory, each copy of ρ used incurs a cost; the rate r then tells us how many copies of ω we can obtain per copy of ρ. In our setting, we instead consider the size of the manipulation protocol Λ_n to be the costly parameter. If the protocols Λ_n were deterministic, the two settings would be exactly the same; however, our approach allows us to take Λ_n to be a probabilistic operation and repeat the protocol until it succeeds, at no extra cost.

It is also of interest to study the *strong converse rates* $r^\dagger(\rho \to \omega)$ and $r^\dagger_{\rm prob}(\rho \to \omega)$, which are defined analogously except that the error, instead of going to zero, is merely constrained to not tend to 1. Precisely,

$$r^\dagger_{\rm prob}(\rho\to\omega) := \sup\left\{ r \,\middle|\, \liminf_{n\to\infty} \inf_{\Lambda_n\in\mathcal{O}} \frac{1}{2}\left\| \frac{\Lambda_n(\rho^{\otimes n})}{\operatorname{Tr}\Lambda_n(\rho^{\otimes n})} - \omega^{\otimes\lfloor rn\rfloor}\right\|_1 < 1 \right\}, \tag{4}$$

and similarly for $r^\dagger$. This gives a threshold for achievable protocols: attempting transformations at any rate higher than the strong converse would necessarily incur a very large (tending to 1) error.

Similar rates were previously studied in the transformations of multipartite entangled pure states [16–18], in which context a surprising connection with algebraic complexity theory was identified. There, however, no error whatsoever was allowed in the transformation. Although this stricter requirement may be suitable for pure states, such a definition cannot be applied to general quantum systems: in the distillation from noisy mixed states, an asymptotically vanishing error *must* be allowed for the transformations to be possible [12,19], and—as we will shortly see—this error cannot vanish faster than exponentially.

C. Quantum divergences

Divergences (relative entropies), typically understood to be entropic distances between density matrices, are a commonly encountered concept in quantum theory [20]. The most fundamental is certainly the quantum relative entropy $D(\rho\|\sigma) = \operatorname{Tr}\rho(\log\rho - \log\sigma)$ [21] itself, where ρ and σ are density matrices, and we take the logarithm to be to the base 2. In resource-theoretic applications, it becomes important to study the optimized divergence $D_{\mathcal{F}}(\rho) := \min_{\sigma\in\mathcal{F}} D(\rho\|\sigma)$ [22,23]. Then, due to the potential nonadditivity of this function [24], in asymptotic settings it is the regularized relative entropy [25]

$$D^\infty_{\mathcal{F}}(\rho) := \lim_{n\to\infty}\frac{1}{n} D_{\mathcal{F}_n}(\rho^{\otimes n}) \tag{5}$$

that finds operational applications.

A different divergence, one that finds use primarily in one-shot settings, is the max-relative entropy [26] defined as $D_{\max}(\rho\|\sigma) := \inf\{\log\lambda \mid \rho \leqslant \lambda\sigma\}$. Defining the optimized max-relative entropy $D_{\max,\mathcal{F}}$ as above, an important aspect of this quantity is that, after "smoothing" and regularizing, it actually yields the regularized relative entropy itself [9,27]:[1]

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\min_{\frac12\|\rho'-\rho^{\otimes n}\|_1\leqslant\varepsilon}\frac{1}{n}D_{\max,\mathcal{F}_n}(\rho') = D^\infty_{\mathcal{F}}(\rho), \tag{6}$$

where ρ' is constrained to be a density matrix.

Another well-known one-shot divergence is the min-relative entropy [26], given by $D_{\min}(\psi\|\sigma) = -\log\langle\psi|\sigma|\psi\rangle$ for a pure state $\psi = |\psi\rangle\langle\psi|$.

III. GENERAL LIMITATIONS ON PROBABILISTIC TRANSFORMATIONS

A. Projective relative entropy

Our approach will require the study of a different type of divergence, which we dub the *projective relative entropy*:

$$\mathbb{D}_\Omega(\rho\|\sigma) := D_{\max}(\rho\|\sigma) + D_{\max}(\sigma\|\rho). \tag{7}$$

This is also known as the Hilbert projective metric between ρ and σ with respect to the positive semidefinite cone [28,29]. The notation $\mathbb{D}$ is used here to avoid confusion with quantities based on the standard relative entropy D.

The optimized variant of this quantity,

$$\mathbb{D}_{\Omega,\mathcal{F}}(\rho) := \min_{\sigma\in\mathcal{F}}\mathbb{D}_\Omega(\rho\|\sigma), \tag{8}$$

which we refer to as the *projective relative entropy of a resource*, is directly related to the projective robustness introduced in [12,30] and used to characterize one-shot transformations in probabilistic settings. Since we are interested in asymptotic state manipulation, let us define the regularization:

$$\mathbb{D}^\infty_{\Omega,\mathcal{F}}(\rho) := \lim_{n\to\infty}\frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}_n}(\rho^{\otimes n}). \tag{9}$$

As with the max-relative entropy, it is also natural to expect the *smoothed* regularization of $\mathbb{D}_\Omega$ to come into play, that is,

[1]The curious reader might wonder whether this result is related to the generalized quantum Stein lemma of [59], in the proof of which some issues were recently identified [43]. Fortunately, the asymptotic equipartition property of $D_{\max}$ that we employ here is independent of that result, as can be seen both in the proof found in [9] and the independent proof in [27].

the quantity

$$\mathbb{D}^{\infty,\bullet}_{\Omega,\mathcal{F}}(\rho) := \lim_{\varepsilon\to 0}\lim_{n\to\infty}\min_{\frac{1}{2}\|\rho'-\rho^{\otimes n}\|_1\leqslant\varepsilon}\frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}_n}(\rho'). \tag{10}$$

We show that this simply equals the regularized relative entropy with respect to the set $\mathcal{F}$ (see Appendix B).

Lemma 1 (*Asymptotic equipartition property for the projective relative entropy*). In every convex resource theory with free states $\mathcal{F}$, it holds that

$$\mathbb{D}^{\infty,\bullet}_{\Omega,\mathcal{F}}(\rho) = D^{\infty}_{\mathcal{F}}(\rho). \tag{11}$$

That is, although $D_{\max}$ and $\mathbb{D}_{\Omega}$ are very differently behaved quantities, they both give rise to the same quantity asymptotically. The above result will greatly simplify the asymptotic bounds on probabilistic state transformations and allow for direct comparisons with the deterministic case.

B. Converse bound

We now use the regularized projective relative entropy to establish a general converse bound on all state transformations in the probabilistic setting.

Proposition 2. For all states ρ and ω, the following holds:

$$r_{\text{prob}}(\rho\to\omega) \leqslant \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D^{\infty}_{\mathcal{F}}(\omega)}. \tag{12}$$

See Appendix C for a proof.

This should be compared with the well-known upper bound on *deterministic* transformation rates, given by [31,32]

$$r(\rho\to\omega) \leqslant \frac{D^{\infty}_{\mathcal{F}}(\rho)}{D^{\infty}_{\mathcal{F}}(\omega)}. \tag{13}$$

An interesting aspect is that both bounds have the regularized relative entropy $D^{\infty}_{\mathcal{F}}(\omega)$ in the denominator, but the numerator is very different—the probabilistic bound in (12) features the regularization of the projective relative entropy, without any smoothing. It is not difficult to find examples of states such that $\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho) > D^{\infty}_{\mathcal{F}}(\rho)$, meaning that the probabilistic upper bound can be strictly larger. As we will see, this is in fact the best bound possible, as it can be achieved exactly in many cases.

A word of caution is necessary here. It may be the case that the quantity $\mathbb{D}_{\Omega,\mathcal{F}}(\rho)$ actually diverges to infinity [30], e.g., when ρ is a pure state. However, the problem is avoided for the more practically relevant, high-rank quantum states, in which case (12) is typically a well-defined and finite bound.

C. Improved bound for distillation

We can obtain a number of improvements to our results when the task of *distillation* (purification) is considered. Here, the target state is chosen to be a pure state ψ, as is often the case in practical state transformations where one aims to purify a noisy system. We first obtain the following improvement over Proposition 2, the proof of which can be found in Appendix D.

Proposition 3. Every sequence $(\Lambda_n)_n$ of distillation protocols satisfies the following tradeoff relation between its rate r and transformation errors $\varepsilon_n := \frac{1}{2}\|\Lambda_n(\rho^{\otimes n}) - \psi^{\otimes\lfloor rn\rfloor}\|_1$:

$$r \leqslant \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D^{\infty}_{\min,\mathcal{F}}(\psi)} - \frac{\limsup_{n\to\infty}\frac{1}{n}\log\varepsilon_n^{-1}}{D^{\infty}_{\min,\mathcal{F}}(\psi)}. \tag{14}$$

In particular, the strong converse rate satisfies

$$r^{\dagger}_{\text{prob}}(\rho\to\psi) \leqslant \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D^{\infty}_{\min,\mathcal{F}}(\psi)}. \tag{15}$$

Here, $D^{\infty}_{\min,\mathcal{F}}(\psi) := \lim_{n\to\infty}\frac{1}{n}D_{\min,\mathcal{F}_n}(\psi^{\otimes n})$. The bound in (14) allows one to understand exactly the rates achievable with a given error sequence $(\varepsilon_n)_n$. One important consequence is that, if ε_n goes to zero too quickly (faster than exponentially), then one *cannot* distill at any nonzero rate; in other words, the result shows that for every distillation protocol, including the most general probabilistic ones, the errors must satisfy $\varepsilon_n = 2^{-O(n)}$.

IV. ACHIEVABILITY RESULTS

A. Affine resource theories

Our general converse bound gives a universal constraint: no matter how many copies of states we have at our disposal, and no matter how many times we repeat a given protocol, the bound of Proposition 2 cannot be exceeded. In order to understand the tightness of this result, it is then of interest to investigate when the bound can be actually achieved, giving an exact expression for the asymptotic transformation rate between any two states. This is the case in the class of *affine* resource theories [33,34], defined such that the set of free states $\mathcal{F}$ is the intersection of some affine subspace of Hermitian operators with the set of all density matrices. This class includes, for instance, the theories of thermodynamics (athermality) [35], coherence [36], asymmetry [37], or imaginarity [38].

Proposition 4. Consider any affine resource theory. Then, for all states ρ and ω, the transformation rate under resource-nongenerating operations $\mathcal{O}$ satisfies

$$r_{\text{prob}}(\rho\to\omega) = \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D^{\infty}_{\mathcal{F}}(\omega)}. \tag{16}$$

See Appendix E for a proof.

This result is more general than known results in the characterization of deterministic state conversion. While the relative entropy upper bound in Eq. (13) is tight in some theories (e.g., athermality [39] or coherence [40]), there is no known result which shows that bound to be tight in general classes of resources.[2]

[2] We remark that [46] claimed to show that the relative entropy bound is tight in almost all resource theories. However, due to issues in the proof of the underlying work [59], the result is not known to be true [43]. Furthermore, the framework of [46] employs, instead of free operations $\mathcal{O}$, a class of operations that is only *approximately* free, and may actually create large amounts of resources in certain cases [42].

B. General resource theories

Going beyond affine resource theories requires a slightly different approach. To this end, consider the *standard robustness* of a given resource [41], $R_{s,\mathcal{F}}(\rho) := \inf\{\lambda \mid \rho + \lambda\sigma \propto \sigma' \in \mathcal{F},\ \sigma \in \mathcal{F}\}$, together with its regularized variant:

$$\mathbb{D}^{\infty,\bullet}_{s,\mathcal{F}}(\rho) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \min_{\frac{1}{2}\|\rho'-\rho^{\otimes n}\|_1 \leqslant \varepsilon} \frac{1}{n} \log\left[1 + R_{s,\mathcal{F}_n}(\rho')\right]. \tag{17}$$

Contrary to the case of $D_{\max,\mathcal{F}}$ in Eq. (6), we do not know how to express this quantity with an alternative formula that does not involve a smoothing. Nevertheless, it can be used to establish a general achievable transformation rate.

Proposition 5. Consider any resource theory such that $R_{s,\mathcal{F}}(\rho) < \infty$ for all states. Then, for all states ρ and ω, the transformation rate under resource-nongenerating operations $\mathcal{O}$ satisfies

$$r_{\text{prob}}(\rho \to \omega) \geqslant \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{\mathbb{D}^{\infty,\bullet}_{s,\mathcal{F}}(\omega)}. \tag{18}$$

A proof can be found in Appendix F.

Note that the achievable rate in Proposition 5 does not always match the converse bound in Proposition 2, as recently an example of a state satisfying $\mathbb{D}^{\infty,\bullet}_{s,\mathcal{F}}(\rho) > D^{\infty}_{\mathcal{F}}(\rho)$ was found [42]. However, we will see that the two quantities can be equal in some relevant cases.

C. Distillation

The task of distillation is a special case for which, in many theories of interest and even for some nonaffine theories, we can evaluate the optimal asymptotic rate exactly because our upper (Proposition 2 or 3) and lower (Proposition 4 or 5) bounds actually coincide. This is because the target state of distillation ψ is often chosen to be a maximally resourceful state (e.g., a Bell state Φ_2 in entanglement theory), the properties of which allow for a simplified quantification of its resources.

Notably, we can exactly evaluate the rate of probabilistic entanglement distillation under nonentangling operations from any state:

$$r_{\text{prob}}(\rho \to \Phi_2) = r^{\dagger}_{\text{prob}}(\rho \to \Phi_2) = \mathbb{D}^{\infty}_{\Omega,\text{SEP}}(\rho), \tag{19}$$

where "SEP" denotes the set of separable states. This can be considered surprising in light of the fact that, for deterministic transformations, the exact rate of distillation under nonentangling operations is not known [9,43].

A similar result can be derived for more general resource theories, and we present a detailed technical discussion of this property in Appendix G.

D. On the probability of success in the asymptotic limit

Many conventional distillation protocols also incorporate probabilistic elements—including even the earliest schemes for entanglement distillation [8]—but in such settings these "probabilistic" rates have been shown to be equivalent to standard deterministic ones [44]. What is the difference between such protocols and the transformations considered here?

To gain some insight into why the standard relative entropy $D^{\infty}_{\mathcal{F}}(\rho)$ is insufficient to characterize the asymptotic rates of probabilistic transformations as considered in our paper, let us look into the optimal probability of success in such protocols. By construction, transformation rates in our framework do not depend on the probability of success of the protocols. This means that our definition of a probabilistic rate r_{prob} allows for a sequence of protocols $(\Lambda_n)_n$ such that the probability of success vanishes in the asymptotic limit, that is, $\lim_{n\to\infty} \operatorname{Tr}\Lambda_n(\rho^{\otimes n}) = 0$. This is because we are effectively disregarding the exact probability, so the rate is not affected by whether $\operatorname{Tr}\Lambda_n(\rho^{\otimes n})$ becomes arbitrarily small.

This is precisely the aspect that distinguishes our approach from conventional asymptotic transformations. To see this, we define a variant of a probabilistic rate where protocols with vanishing success probability are not allowed:

$$r_{\text{prob}>0}(\rho \to \omega) := \sup\left\{ r \,\middle|\, (\Lambda_n)_n \in \mathcal{O},\ \liminf_{n\to\infty} \operatorname{Tr}\Lambda_n(\rho^{\otimes n}) > 0,\ \lim_{n\to\infty} \frac{1}{2}\left\| \frac{\Lambda_n(\rho^{\otimes n})}{\operatorname{Tr}\Lambda_n(\rho^{\otimes n})} - \omega^{\otimes\lfloor rn\rfloor} \right\|_1 = 0 \right\}. \tag{20}$$

Our asymptotic equipartition property of Lemma 1 then allows us to show that the deterministic upper bound based on the relative entropy $D^{\infty}_{\mathcal{F}}$ cannot be exceeded by such protocols.

Proposition 6. The rate of any state transformation with a probability of success that does not asymptotically vanish satisfies

$$r_{\text{prob}>0}(\rho \to \omega) \leqslant \frac{D^{\infty}_{\mathcal{F}}(\rho)}{D^{\infty}_{\mathcal{F}}(\omega)}. \tag{21}$$

Proposition 6 shows that a vanishing probability of success is, in many cases, *required* to gain an advantage over probabilistic protocols in the asymptotic limit. That is because the rate $\frac{D^{\infty}_{\mathcal{F}}(\rho)}{D^{\infty}_{\mathcal{F}}(\omega)}$ is achievable deterministically in many known settings, and indeed it has been conjectured to be achievable in general resource theories under so-called asymptotically resource-nongenerating operations [9,43,46]. Whenever this is the case, we have that the two rates are equal, $r_{\text{prob}>0}(\rho \to \omega) = r(\rho \to \omega)$, while the rate $r_{\text{prob}}(\rho \to \omega)$ considered in our paper is typically strictly larger (see, e.g., the upcoming Sec. V A).

Remarkably, standard techniques leveraging the asymptotic continuity and the "not-too-convexity" of $D_{\mathcal{F}}$ [45] do not suffice to prove the above result—they only succeed in establishing the weaker statement that the *deterministic* rate is upper bounded by the right-hand side of (21). Our proof, instead, makes essential use of the asymptotic equipartition property for $D_{\Omega,\mathcal{F}}$ established in Lemma 1. For further details on this point as well as a full proof of the result, see Appendix H.

V. EXAMPLES

A. Isotropic entangled states

Isotropic states are representative examples of entangled states that enjoy simplified entanglement properties due to their strong symmetry [48]. For a local dimension d, they are defined as

$$\rho_p := p\Phi_d + (1-p)\frac{\mathbb{1} - \Phi_d}{d^2-1} \tag{22}$$

where Φ_d is a two-qudit maximally entangled state. The set of free states in this resource theory is given by all separable states, $\mathcal{F} = \text{SEP}$ (One could alternatively consider the resource theory where $\mathcal{F}$ is given by all states with a positive partial transpose; the results below remain unchanged.).

We can use our results to exactly evaluate the rate of probabilistic entanglement distillation $r_{\text{prob}}(\rho_p \to \Phi_2)$: assuming that $p \geqslant 1/d$ (as otherwise ρ_p is separable [48]), we find that

$$r_{\text{prob}}(\rho_p \to \Phi_2) = \mathbb{D}^{\infty}_{\Omega,\text{SEP}}(\rho_p) = \mathbb{D}_{\Omega,\text{SEP}}(\rho_p) = \log\frac{p(d-1)}{1-p}, \tag{23}$$

while any rate achievable under deterministic protocols satisfies [1,9,49]

$$r^{\dagger}(\rho_p \to \Phi_2) \leqslant D^{\infty}_{\text{SEP}}(\rho_p)$$
$$= p\log d + (1-p)\log\frac{d}{d-1} - h_2(p), \tag{24}$$

where h_2 is the binary entropy function, with the bound conjectured to be tight [9,43]. See Appendix I for a proof. The gap between the two quantities is depicted in Fig. 2, showing that probabilistic asymptotic protocols exhibit prominently higher rates than purely deterministic ones.

B. Dichotomies and distinguishability

The case of transformations of pairs of quantum states, i.e., finding a channel which satisfies $\Lambda(\rho_1) = \omega_1$ and $\Lambda(\rho_2) = \omega_2$, has been studied under the name of quantum dichotomies [6] and underlies the resource theory of asymmetric distinguishability [5,7]. Asymptotic transformations here are studied in the sense that the transformation $\rho_1^{\otimes n} \to \omega_1^{\otimes \lfloor rn \rfloor}$ may be realized approximately (as in the definition of the asymptotic rate), but $\rho_2^{\otimes n} \to \omega_2^{\otimes \lfloor rn \rfloor}$ must always be exact. It was then shown that the deterministic rate of such transformations is given by $D(\rho_1\|\rho_2)/D(\omega_1\|\omega_2)$ [6,7]. We get an analogous result also in our case: the probabilistic conversion rate is exactly $\mathbb{D}_{\Omega}(\rho_1\|\rho_2)/D(\omega_1\|\omega_2)$, which is typically strictly larger than the deterministic rate. A special case of this task has been further studied in [50] and interpreted therein as a postselected variant of quantum hypothesis testing.

This result can also be applied to resource theories with only a single free state, such as the resource theory of athermality with Gibbs-preserving operations or the resource theory of purity [51].

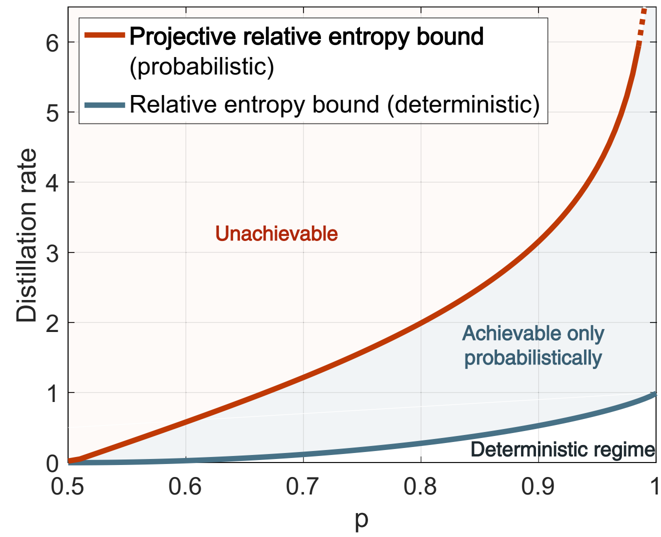

FIG. 2. Entanglement distillation from two-qubit isotropic states. We plot the most general upper bound on the rate of deterministic entanglement distillation under all nonentangling protocols, the regularized relative entropy $D^{\infty}_{\text{SEP}}(\rho_p)$, and compare it with the exact achievable rate of probabilistic entanglement distillation under nonentangling protocols, namely, the regularized projective relative entropy $\mathbb{D}^{\infty}_{\Omega,\text{SEP}}(\rho_p)$. It can be seen that, for all values of $p > 0.5$, allowing nondeterministic protocols leads to significantly higher distillation rates. We also observe that the probabilistic rate becomes unbounded as $p \to 1$, which is consistent with the fact that all pure states can be probabilistically interconverted with nonentangling operations [30,47]. The local dimension is chosen to be $d = 2$.

VI. DISCUSSION

We introduced a framework for the asymptotic manipulation of quantum states with probabilistic protocols and established comprehensive methods for its characterization. We specifically used a class of resource monotones based on the projective relative entropy $\mathbb{D}_{\Omega}$ to establish general upper and lower bounds for transformation rates, showing them to be tight and exactly computable in relevant cases.

There are two main facts revealed by the results of our paper. *A priori*, it is not even obvious if probabilistic rates are constrained at all, and even if so, what these constraints may be—if we are allowing one to repeat the transformation protocols an unbounded number of times, could it not be feasible that *every* transformation becomes eventually possible? We showed that not to be the case, revealing general limitations that every probabilistic state transformation protocol is subject to. On the other hand, we saw that probabilistic protocols can (and generally do) outperform deterministic ones, showing that the framework considered in our work does indeed exceed the capabilities of standard quantum Shannon theory.

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APPENDIX A: NOTATION AND BASIC PROPERTIES

1. Projective relative entropy

In Table I we review the definitions of the various divergences and their regularized forms.

We will sometimes use the notation $\Omega_{\mathcal{F}}$ for the nonlogarithmic variant of the projective relative entropy $\mathbb{D}_{\Omega,\mathcal{F}}$, namely, $\Omega_{\mathcal{F}}(\rho) = 2^{\mathbb{D}_{\Omega,\mathcal{F}}(\rho)}$.

We recall some of the most important properties of $\mathbb{D}_{\Omega,\mathcal{F}}$ [30].

(i) It is monotonic under any probabilistic transformation protocol: for all $\Lambda \in \mathcal{O}$, $\mathbb{D}_{\Omega,\mathcal{F}}(\frac{\Lambda(\rho)}{\operatorname{Tr}\Lambda(\rho)}) \leqslant \mathbb{D}_{\Omega,\mathcal{F}}(\rho)$.

(ii) It is faithful, i.e., $\mathbb{D}_{\Omega,\mathcal{F}}(\rho) = 0$ iff $\rho \in \mathcal{F}$.

(iii) It is invariant under scaling, i.e., $\mathbb{D}_{\Omega,\mathcal{F}}(\lambda\rho) = \mathbb{D}_{\Omega,\mathcal{F}}(\rho)$ for all $\lambda > 0$.

(iv) It may diverge: It is finite, i.e., $\mathbb{D}_{\Omega,\mathcal{F}}(\rho) < \infty$, if and only if there exists a free state $\sigma \in \mathcal{F}$ such that $\operatorname{supp}\rho = \operatorname{supp}\sigma$.

(v) It is lower semicontinuous, quasiconvex,[3] and subadditive under tensor products.

(vi) It can be written as a convex optimization problem:

$$\begin{aligned}\Omega_{\mathcal{F}}(\rho) &= \inf\{\gamma \in \mathbb{R} \mid \rho \leqslant \widetilde{\sigma} \leqslant \gamma\rho,\ \widetilde{\sigma} \in \operatorname{cone}(\mathcal{F})\}\\ &= \sup\left\{\operatorname{Tr}A\rho \,\middle|\, \operatorname{Tr}B\rho = 1,\ B - A \in \operatorname{cone}(\mathcal{F})^*\right.\\ &\qquad\left. A, B \geqslant 0\right\}\\ &= \sup\left\{\frac{\operatorname{Tr}A\rho}{\operatorname{Tr}B\rho} \,\middle|\, \frac{\operatorname{Tr}A\sigma}{\operatorname{Tr}B\sigma} \leqslant 1\ \forall\sigma \in \mathcal{F},\ A, B \geqslant 0\right\},\end{aligned} \tag{A1}$$

where $\operatorname{cone}(\mathcal{F})$ denotes the convex cone induced by the set of free states $\mathcal{F}$ and $\operatorname{cone}(\mathcal{F})^*$ is its dual cone [52].

Let us briefly comment on property (iv), that is, the fact that $\mathbb{D}_{\Omega,\mathcal{F}}$ might take infinite values. This potential issue is always avoided for highly mixed states—indeed, Axiom II of Sec. II A ensures that there exists a full-rank free state, so $\mathbb{D}_{\Omega,\mathcal{F}}(\rho) < \infty$ for all full-rank ρ. Extended variants of this property can be shown in specific resource theories; for example, in the theory of entanglement, we can show the following.

Lemma A1. Consider the resource theory of bipartite entanglement with local systems of dimensions d_A and d_B. Then, every state ρ the rank of which satisfies $\operatorname{rank}(\rho) \geqslant d_A d_B - 1$ is such that $\mathbb{D}_{\Omega,\mathrm{SEP}}(\rho) < \infty$.

Proof. The full-rank case follows directly from Axiom II, so assume that $\operatorname{rank}(\rho) = d_A d_B - 1$. The projection onto the

TABLE I. Summary of the definitions of the different divergences studied in this paper. We remark that the minima over $\mathcal{F}$ are achieved due to the lower semicontinuity of the involved quantities and the assumed compactness of $\mathcal{F}$. The exception is $\mathbb{D}_{s,\mathcal{F}}$, which may be unbounded depending on the resource theory.

Quantity	Name	Definition	Remarks
$D_{\mathcal{F}}(\rho)$	Relative entropy of resource	$\min_{\sigma\in\mathcal{F}} D(\rho\|\sigma) = \min_{\sigma\in\mathcal{F}} \operatorname{Tr}\rho(\log\rho - \log\sigma)$	
$D_{\max,\mathcal{F}}(\rho)$	Max-relative entropy of resource	$\min_{\sigma\in\mathcal{F}} D_{\max}(\rho\|\sigma) = \min_{\sigma\in\mathcal{F},\lambda\in\mathbb{R}_+}\{\log\lambda \mid \rho \leqslant \lambda\sigma\}$	
$D_{\min,\mathcal{F}}(\rho)$	Min-relative entropy of resource	$\min_{\sigma\in\mathcal{F}} D_{\min}(\rho\|\sigma) = \min_{\sigma\in\mathcal{F}}[-\log(\operatorname{Tr}\Pi_\rho\sigma)]$	
$\mathbb{D}_{\Omega,\mathcal{F}}(\rho)$	Projective relative entropy of resource	$\min_{\sigma\in\mathcal{F}} \mathbb{D}_\Omega(\rho\|\sigma) = \min_{\sigma\in\mathcal{F}}[D_{\max}(\rho\|\sigma) + D_{\max}(\sigma\|\rho)]$	
$\mathbb{D}_{s,\mathcal{F}}(\rho)$	Logarithmic standard robustness of resource	$\log[1 + R_{s,\mathcal{F}}(\rho)] = \inf_{\sigma,\sigma'\in\mathcal{F},\lambda\in\mathbb{R}_+}\{\log\lambda \mid \rho = \lambda\sigma - (\lambda-1)\sigma'\}$	
$D^\infty_{\mathcal{F}}(\rho)$	Regularized relative entropy of resource	$\lim_{n\to\infty} \frac{1}{n} D_{\mathcal{F}_n}(\rho^{\otimes n})$	
$\mathbb{D}^\infty_{\Omega,\mathcal{F}}(\rho)$	Regularized projective relative entropy of resource	$\lim_{n\to\infty} \frac{1}{n} \mathbb{D}_{\Omega,\mathcal{F}_n}(\rho^{\otimes n})$	
$D^{\infty,\bullet}_{\max,\mathcal{F}}(\rho)$	Smoothed regularized max-relative entropy	$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \min_{\frac{1}{2}\|\rho'-\rho^{\otimes n}\|_1\leqslant\varepsilon} \frac{1}{n} D_{\max,\mathcal{F}_n}(\rho')$	$= D^\infty_{\mathcal{F}}(\rho)$ [9,27]
$\mathbb{D}^{\infty,\bullet}_{\Omega,\mathcal{F}}(\rho)$	Smoothed regularized projective relative entropy	$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \min_{\frac{1}{2}\|\rho'-\rho^{\otimes n}\|_1\leqslant\varepsilon} \frac{1}{n} \mathbb{D}_{\Omega,\mathcal{F}_n}(\rho')$	$= D^\infty_{\mathcal{F}}(\rho)$ (Lemma 1)
$\mathbb{D}^{\infty,\bullet}_{s,\mathcal{F}}(\rho)$	Smoothed regularized standard robustness	$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} \min_{\frac{1}{2}\|\rho'-\rho^{\otimes n}\|_1\leqslant\varepsilon} \frac{1}{n} \mathbb{D}_{s,\mathcal{F}_n}(\rho')$	$\neq D^\infty_{\mathcal{F}}(\rho)$ in general [42]

[3]We remind the reader that a function $f : C \to \mathbb{R} \cup \{+\infty\}$ defined on a convex set C is said to be quasiconvex if $f(tx + (1-t)y) \leqslant \max\{f(x), f(y)\}$ holds for all $x, y \in C$ and $t \in [0, 1]$.

support of ρ is then given by $\Pi_\rho = \mathbb{1} - |\phi\rangle\langle\phi|$ for some pure $|\phi\rangle$. Every such Π_ρ is known to be a separable operator [53]. But $\lambda_{\min}(\rho)\Pi_\rho \leqslant \rho \leqslant \lambda_{\max}(\rho)\Pi_\rho$ where $\lambda_{\min}$ and $\lambda_{\max}$ denote the smallest and largest nonzero eigenvalue, respectively, meaning that $\mathbb{D}_{\Omega,\mathrm{SEP}}(\rho) \leqslant \log\lambda_{\max}(\rho) - \log\lambda_{\min}(\rho) < \infty$. ■

For lower-rank states, it appears difficult to obtain general statements, and the verification of the finiteness of $\mathbb{D}_{\Omega,\mathcal{F}}$ needs to be performed on a state-by-state basis. Nevertheless, the results of this paper can be readily applied to any state that can be shown to have a finite value of $\mathbb{D}_{\Omega,\mathcal{F}}$.

Note also that Axioms III and IV guarantee that $\mathbb{D}_{\Omega,\mathcal{F}}(\rho) < \infty \iff \mathbb{D}^\infty_{\Omega,\mathcal{F}}(\rho) < \infty$.

2. Regularization

We often use some well-known facts about asymptotic regularizations of functions, which we collect below.

Fact 1 (Fekete's lemma [54]*).* Let f be a weakly subadditive function of density matrices, that is, one such that $f(\rho^{\otimes m} \otimes \rho^{\otimes n}) \leqslant f(\rho^{\otimes m}) + f(\rho^{\otimes n})$ for all states ρ and for all $m, n \in \mathbb{N}$. Then, the regularization $f^\infty(\rho) := \lim_{n\to\infty} \frac{1}{n} f(\rho^{\otimes n})$ exists and equals $\inf_n \frac{1}{n} f(\rho^{\otimes n})$.

Fact 2 (Weak additivity of regularization [25]*).* Let f be a function such that the regularization f^∞ exists. Then, f^∞ is weakly additive, that is, $f^\infty(\rho^{\otimes n}) = n f^\infty(\rho)$.

APPENDIX B: ASYMPTOTIC EQUIPARTITION PROPERTY

Lemma 1. For every sequence of sets $(\mathcal{F}_n)_n$ satisfying Axioms I–IV, the smoothed regularization of the projective relative entropy is simply the regularized relative entropy of the given resource. That is,

$$\mathbb{D}^{\infty,\bullet}_{\Omega,\mathcal{F}}(\rho) = D^\infty_{\mathcal{F}}(\rho) \tag{B1}$$

where we recall that

$$\mathbb{D}^{\infty,\bullet}_{\Omega,\mathcal{F}}(\rho) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \min_{\frac{1}{2}\|\rho'-\rho^{\otimes n}\|_1 \leqslant \varepsilon} \frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}_n}(\rho') \tag{B2}$$

with the minimization over normalized quantum states ρ'.

Proof. It is already known that, for every sequence of sets $(\mathcal{F}_n)_n$ satisfying Axioms I–IV, the smoothed regularization of $D_{\max}$ is precisely the regularized relative entropy $D^\infty_{\mathcal{F}}$ [9,27]; specifically,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \min_{\frac{1}{2}\|\rho'-\rho^{\otimes n}\|_1 \leqslant \varepsilon} \frac{1}{n} D_{\max,\mathcal{F}_n}(\rho') = D^\infty_{\mathcal{F}}(\rho). \tag{B3}$$

Since $\mathbb{D}_{\Omega,\mathcal{F}}(\rho) \geqslant D_{\max,\mathcal{F}}(\rho)$ by definition, this immediately gives that

$$\mathbb{D}^{\infty,\bullet}_{\Omega,\mathcal{F}}(\rho) \geqslant D^\infty_{\mathcal{F}}(\rho), \tag{B4}$$

and so it suffices to show the opposite inequality. We will proceed to show that the term $D_{\max}(\sigma\|\rho)$ that distinguished $\mathbb{D}_\Omega$ from $D_{\max}$ becomes asymptotically irrelevant, and the two divergences must converge to the same value.

To this end, fix $\delta > 0$; for all sufficiently small $\varepsilon > 0$, there exists a sequence of states $(\rho'_n)_n$ with $\frac{1}{2}\|\rho'_n - \rho^{\otimes n}\|_1 \leqslant \frac{\varepsilon}{2}$ which satisfies

$$\left| \lim_{n\to\infty} \frac{1}{n} \min_{\sigma\in\mathcal{F}_n} D_{\max}(\rho'_n\|\sigma) - D^\infty_{\mathcal{F}}(\rho) \right| \leqslant \delta. \tag{B5}$$

For every such choice of $(\rho'_n)_n$, let $(\sigma_n)_n$ be a sequence of states such that $\min_{\sigma\in\mathcal{F}_n} D_{\max}(\rho'_n\|\sigma) = D_{\max}(\rho'_n\|\sigma_n) =: \lambda_n \ \forall n$. Define

$$\omega_n := \frac{2^{-\eta}\sigma_n + \rho'_n}{1 + 2^{-\eta}}, \tag{B6}$$

where we fix $\eta := \log(2\varepsilon^{-1} - 1)$, so that

$$\begin{aligned}\frac{1}{2}\|\omega_n - \rho'_n\|_1 &= \frac{1}{2}\left\| \frac{2^{-\eta}\sigma_n + \rho'_n}{1 + 2^{-\eta}} - \frac{(1 + 2^{-\eta})\rho'_n}{1 + 2^{-\eta}} \right\|_1 \\ &= \frac{1}{2}\frac{2^{-\eta}}{1 + 2^{-\eta}}\|\sigma_n - \rho'_n\|_1 \\ &\leqslant \frac{2^{-\eta}}{1 + 2^{-\eta}} \\ &= \frac{\varepsilon}{2}.\end{aligned} \tag{B7}$$

By construction, we have that

$$\omega_n \leqslant \frac{2^{-\eta}\sigma_n + 2^{D_{\max}(\rho'_n\|\sigma_n)}\sigma_n}{1 + 2^{-\eta}} = \frac{2^{-\eta} + 2^{\lambda_n}}{1 + 2^{-\eta}}\sigma_n \tag{B8}$$

and we additionally observe that

$$\sigma_n \leqslant \sigma_n + 2^\eta\rho'_n = (1 + 2^{-\eta})\frac{\sigma_n + 2^\eta\rho'_n}{1 + 2^{-\eta}} = (2^\eta + 1)\,\omega_n. \tag{B9}$$

Thus

$$\begin{aligned}\frac{1}{n}\mathbb{D}_\Omega(\omega_n\|\sigma_n) &\leqslant \frac{1}{n}\log(2^{-\eta} + 2^{\lambda_n}) - \frac{1}{n}\log(1 + 2^{-\eta}) \\ &\quad + \frac{1}{n}\log(2^\eta + 1) \\ &\leqslant \frac{1}{n}\log(1 + 2^{-\eta}) + \frac{1}{n}\lambda_n - \frac{1}{n}\log(1 + 2^{-\eta}) \\ &\quad + \frac{1}{n}\log(2^\eta + 1) \\ &= \frac{1}{n}\lambda_n - \frac{1}{n}\log\frac{\varepsilon}{2}.\end{aligned} \tag{B10}$$

Taking the limit then gives

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{D}_\Omega(\omega_n\|\sigma_n) \leqslant \lim_{n\to\infty} \frac{1}{n}\lambda_n \leqslant D^\infty_{\mathcal{F}}(\rho) + \delta. \tag{B11}$$

Altogether we have shown that, having fixed $\delta > 0$, for all sufficiently small $\varepsilon > 0$ we can find a sequence of states $(\omega_n)_n$ such that

$$\tfrac{1}{2}\|\omega_n - \rho^{\otimes n}\|_1 \leqslant \tfrac{1}{2}\|\omega_n - \rho'_n\|_1 + \tfrac{1}{2}\|\rho'_n - \rho^{\otimes n}\|_1 \leqslant \varepsilon \tag{B12}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \min_{\sigma\in\mathcal{F}_n} \mathbb{D}_\Omega(\omega_n\|\sigma) \leqslant D^\infty_{\mathcal{F}}(\rho) + \delta. \tag{B13}$$

Since this holds for all sufficiently small $\varepsilon > 0$, we can take the limit and conclude that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \min_{\frac{1}{2}\|\rho'-\rho^{\otimes n}\|_1 \leqslant \varepsilon} \frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}_n}(\rho') \leqslant D^\infty_{\mathcal{F}}(\rho) + \delta \tag{B14}$$

for all $\delta > 0$, which implies the claim once one takes $\delta \to 0$. ■

A stronger ("strong converse") variant of the above result can be shown for the case when the set $\mathcal{F}$ consists of a single state, with $\mathcal{F}_n = \{\sigma^{\otimes n}\}$. Such a case can be encountered, e.g., in the transformations of quantum dichotomies or in the resource theories of athermality and purity.

Lemma A2. For all $\varepsilon \in (0, 1)$,

$$\lim_{n\to\infty} \min_{\frac{1}{2}\|\rho'-\rho^{\otimes n}\|_1\leqslant\varepsilon} \frac{1}{n}\mathbb{D}_\Omega(\rho'\|\sigma^{\otimes n}) = D(\rho\|\sigma). \tag{B15}$$

Proof. The crucial difference now is that

$$\lim_{n\to\infty} \min_{\frac{1}{2}\|\rho'-\rho^{\otimes n}\|_1\leqslant\varepsilon} \frac{1}{n}D_{\max}(\rho'\|\sigma^{\otimes n}) = D(\rho\|\sigma) \tag{B16}$$

holds for all $\varepsilon \in (0, 1)$ [55,56], which is directly related to the strong converse property of quantum hypothesis testing. The claim then follows from the simple chain of inequalities:

$$\begin{aligned} D(\rho\|\sigma) &= \lim_{n\to\infty} \min_{\frac{1}{2}\|\rho'-\rho^{\otimes n}\|_1\leqslant\varepsilon} \frac{1}{n}D_{\max}(\rho'\|\sigma^{\otimes n}) \\ &\leqslant \lim_{n\to\infty} \min_{\frac{1}{2}\|\rho'-\rho^{\otimes n}\|_1\leqslant\varepsilon} \frac{1}{n}\mathbb{D}_\Omega(\rho'\|\sigma^{\otimes n}) \\ &\overset{\text{(i)}}{\leqslant} \lim_{\varepsilon\to0}\lim_{n\to\infty} \min_{\frac{1}{2}\|\rho'-\rho^{\otimes n}\|_1\leqslant\varepsilon} \frac{1}{n}\mathbb{D}_\Omega(\rho'\|\sigma^{\otimes n}) \\ &\overset{\text{(ii)}}{=} D(\rho\|\sigma). \end{aligned} \tag{B17}$$

Here, (i) descends from the observation that making ε smaller can only increase the minimum over the ε ball, and (ii) comes from Lemma 1. ■

APPENDIX C: GENERAL CONVERSE

Proposition 2. For all states ρ and ω such that $D^\infty_{\mathcal{F}}(\omega) > 0$,

$$r_{\text{prob}}(\rho \to \omega) \leqslant \frac{\mathbb{D}^\infty_{\Omega,\mathcal{F}}(\rho)}{\mathbb{D}^{\infty,\bullet}_{\Omega,\mathcal{F}}(\omega)} = \frac{\mathbb{D}^\infty_{\Omega,\mathcal{F}}(\rho)}{D^\infty_{\mathcal{F}}(\omega)}. \tag{C1}$$

If the set $\mathcal{F}$ consists of a single state, then the above is actually a strong converse bound; specifically, if $\mathcal{F}_n = \{\sigma^{\otimes n}\}$ in the input space and $\mathcal{F}_n = \{\sigma'^{\otimes n}\}$ in the output space, then we get

$$r^\dagger_{\text{prob}}(\rho \to \omega) \leqslant \frac{\mathbb{D}_\Omega(\rho\|\sigma)}{D(\omega\|\sigma')}. \tag{C2}$$

Proof. Suppose that r is an achievable rate; that is, there exists a sequence of protocols $(\Lambda_n)_n \in \mathcal{O}$ such that $\frac{\Lambda_n(\rho^{\otimes n})}{\operatorname{Tr}\Lambda_n(\rho^{\otimes n})} = \tau_n$ with $\frac{1}{2}\|\tau_n - \omega^{\otimes\lfloor rn\rfloor}\|_1 =: \varepsilon_n \to 0$ as $n \to \infty$. Using the monotonicity of $\mathbb{D}_{\Omega,\mathcal{F}}$ under every free probabilistic protocol, we get

$$\mathbb{D}_{\Omega,\mathcal{F}_n}(\rho^{\otimes n}) \geqslant \mathbb{D}_{\Omega,\mathcal{F}_{\lfloor rn\rfloor}}(\tau_n) \geqslant \min_{\frac{1}{2}\|\omega'-\omega^{\otimes\lfloor rn\rfloor}\|_1\leqslant\varepsilon_n} \mathbb{D}_{\Omega,\mathcal{F}_{\lfloor rn\rfloor}}(\omega'). \tag{C3}$$

Then

$$\begin{aligned} \lim_{n\to\infty}\frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}}(\rho^{\otimes n}) &\geqslant \lim_{n\to\infty} \min_{\frac{1}{2}\|\omega'-\omega^{\otimes\lfloor rn\rfloor}\|_1\leqslant\varepsilon_n} \frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}_{\lfloor rn\rfloor}}(\omega') \\ &\geqslant \lim_{\varepsilon\to0}\lim_{n\to\infty} \min_{\frac{1}{2}\|\omega'-\omega^{\otimes\lfloor rn\rfloor}\|_1\leqslant\varepsilon} \frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}_{\lfloor rn\rfloor}}(\omega') \\ &= r\,D^\infty_{\mathcal{F}}(\omega), \end{aligned} \tag{C4}$$

where in the last line we used Lemma 1 and the consequent fact that

$$\begin{aligned} D^\infty_{\mathcal{F}}(\omega) &= \lim_{\varepsilon\to0}\lim_{n\to\infty} \min_{\frac{1}{2}\|\omega'-\omega^{\otimes\lfloor rn\rfloor}\|_1\leqslant\varepsilon} \frac{1}{\lfloor rn\rfloor}\mathbb{D}_{\Omega,\mathcal{F}_{\lfloor rn\rfloor}}(\omega') \\ &= \lim_{\varepsilon\to0}\lim_{n\to\infty} \frac{n}{\lfloor rn\rfloor} \min_{\frac{1}{2}\|\omega'-\omega^{\otimes\lfloor rn\rfloor}\|_1\leqslant\varepsilon} \frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}_{\lfloor rn\rfloor}}(\omega') \\ &= \frac{1}{r}\lim_{\varepsilon\to0}\lim_{n\to\infty} \min_{\frac{1}{2}\|\omega'-\omega^{\otimes\lfloor rn\rfloor}\|_1\leqslant\varepsilon} \frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}_{\lfloor rn\rfloor}}(\omega'). \end{aligned} \tag{C5}$$

If $\mathcal{F}_n = \{\sigma^{\otimes n}\}$, then we no longer need to ensure that $\varepsilon_n \to 0$, as it suffices to take any error $\varepsilon_n \to \varepsilon' < 1$ and invoke Lemma A2. ■

Here we remark that the assumption $D^\infty_{\mathcal{F}}(\omega) > 0$ might seem to be trivially satisfied, but it is not always the case—there can indeed exist resource theories where the regularized relative entropy vanishes for some $\omega \notin \mathcal{F}$, e.g., the theory of asymmetry [57]. Nevertheless, it is true that $D^\infty_{\mathcal{F}}$ can be ensured to be nonzero for all resourceful states in the majority of practically relevant theories, e.g., entanglement [58], making the assumption always satisfied.

APPENDIX D: STRONG CONVERSE FOR DISTILLATION

The result below concerns the case when the target state of the transformation is pure.

We will use the $D_{\min}$ relative entropy to the free states, for which we note that

$$D^\infty_{\min,\mathcal{F}}(\psi) = \lim_{n\to\infty}\frac{1}{n}\log\max_{\sigma\in\mathcal{F}_n}\langle\psi^{\otimes n}|\sigma|\psi^{\otimes n}\rangle^{-1}. \tag{D1}$$

Proposition 3. Consider a pure target state $\omega = \psi \notin \mathcal{F}$. Every physical sequence of distillation protocols $(\Lambda_n)_n$ satisfies the following tradeoff relation between its rate r and transformation errors $\varepsilon_n := \frac{1}{2}\|\Lambda_n(\rho^{\otimes n}) - \psi^{\otimes\lfloor rn\rfloor}\|_1$:

$$r + \frac{\limsup_{n\to\infty}\frac{1}{n}\log\left(\varepsilon_n^{-1}-1\right)}{D^\infty_{\min,\mathcal{F}}(\psi)} \leqslant \frac{\mathbb{D}^\infty_{\Omega,\mathcal{F}}(\rho)}{D^\infty_{\min,\mathcal{F}}(\psi)}. \tag{D2}$$

In particular,

$$r^\dagger_{\text{prob}}(\rho \to \psi) \leqslant \frac{\mathbb{D}^\infty_{\Omega,\mathcal{F}}(\rho)}{D^\infty_{\min,\mathcal{F}}(\psi)}. \tag{D3}$$

Let us remark that, unless the given protocol trivializes with $\lim_{n\to\infty}\varepsilon_n=1$, then it holds that $\limsup_{n\to\infty}\frac{1}{n}\log(\varepsilon_n^{-1}-1) = \limsup_{n\to\infty}\frac{1}{n}\log\varepsilon_n^{-1}$.

Proof. If $\mathbb{D}^\infty_{\Omega,\mathcal{F}}(\rho) = \infty$, then the result is trivial, so assume otherwise. Assume now that there exists a sequence of protocols $(\Lambda_n)_n \in \mathcal{O}$ such that $\frac{\Lambda_n(\rho^{\otimes n})}{\operatorname{Tr}\Lambda_n(\rho^{\otimes n})} = \tau_n$ with error $\varepsilon_n := \frac{1}{2}\|\tau_n - \psi^{\otimes\lfloor rn\rfloor}\|_1$. We use $\delta_n := 1 - \langle\psi^{\otimes\lfloor rn\rfloor}|\tau_n|\psi^{\otimes\lfloor rn\rfloor}\rangle$ to denote the error in fidelity rather than trace distance.

The result of Theorem 9 in [30] tells us that for each n we necessarily have that

$$\Omega_{\mathcal{F}_n}(\rho^{\otimes n}) \geqslant \frac{(1-\delta_n)[1-F_{\mathcal{F}_{\lfloor rn\rfloor}}(\psi^{\otimes\lfloor rn\rfloor})]}{\delta_n F_{\mathcal{F}_{\lfloor rn\rfloor}}(\psi^{\otimes\lfloor rn\rfloor})}, \tag{D4}$$

where we have defined

$$F_{\mathcal{F}}(\psi) := 2^{-D_{\min,\mathcal{F}}(\psi)} = \max_{\sigma\in\mathcal{F}}\langle\psi|\sigma|\psi\rangle \tag{D5}$$

for simplicity. Equivalently, we have that

$$\left[F_{\mathcal{F}_{\lfloor rn\rfloor}}(\psi^{\otimes\lfloor rn\rfloor})^{-1}-1\right]\leqslant\left(\delta_n^{-1}-1\right)^{-1}\Omega_{\mathcal{F}_n}(\rho^{\otimes n})\leqslant\left(\varepsilon_n^{-1}-1\right)^{-1}\Omega_{\mathcal{F}_n}(\rho^{\otimes n}),\tag{D6}$$

where $\delta_n\leqslant\varepsilon_n$ is a consequence of the tighter Fuchs–van de Graaf inequality $1-F(\rho,\psi)\leqslant\frac{1}{2}\|\rho-\psi\|_1$. Taking the logarithm of the above and dividing by n gives

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n}\log[F_{\mathcal{F}_{\lfloor rn\rfloor}}(\psi^{\otimes\lfloor rn\rfloor})^{-1}-1]&\leqslant\liminf_{n\to\infty}\left[\frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}}(\rho^{\otimes n})-\frac{1}{n}\log\left(\varepsilon_n^{-1}-1\right)\right]\\&\leqslant\limsup_{n\to\infty}\left[\frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}}(\rho^{\otimes n})\right]-\limsup_{n\to\infty}\frac{1}{n}\log\left(\varepsilon_n^{-1}-1\right).\end{aligned}\tag{D7}$$

Here, in the second line we used the fact that $\liminf_{n\to\infty}(a_n-b_n)\leqslant\limsup_{n\to\infty}a_n-\limsup_{n\to\infty}b_n$, as one sees immediately by picking a subsequence $(a_{n_k}-b_{n_k})_k$ with the property that $\lim_{k\to\infty}b_{n_k}=\limsup_{n\to\infty}b_n$. Now, using the fact that $D^\infty_{\min,\mathcal{F}}(\psi)$ is well defined (due to the subadditivity of $D_{\min,\mathcal{F}}$) and weakly additive, we have that the left-hand side reduces to

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n}\log\left[F_{\mathcal{F}_{\lfloor rn\rfloor}}(\psi^{\otimes\lfloor rn\rfloor})^{-1}-1\right]&=\lim_{n\to\infty}\frac{1}{n}\log F_{\mathcal{F}_{\lfloor rn\rfloor}}(\psi^{\otimes\lfloor rn\rfloor})^{-1}\\&=r\,D^\infty_{\min,\mathcal{F}}(\psi),\end{aligned}\tag{D8}$$

which concludes the first part of the proof.

To obtain a strong converse bound, we see that assuming that $\liminf_{n\to\infty}\varepsilon_n\in[0,1)$ entails that $\limsup\varepsilon_n^{-1}>1$, and hence

$$r\,D^\infty_{\min,\mathcal{F}}(\psi)\leqslant\limsup_{n\to\infty}\frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}_n}(\rho^{\otimes n})\tag{D9}$$

as was to be shown. ■

Note that in Proposition 3 we did not need to assume that $\limsup_{n\to\infty}\frac{1}{n}\log(\varepsilon_n^{-1}-1)<\infty$: this is guaranteed by (D6) coupled with the submultiplicativity of $\Omega_{\mathcal{F}}$, which together ensure that ε_n^{-1} is upper bounded by

$$\varepsilon_n^{-1}\leqslant\left[\Omega_{\mathcal{F}}(\rho)\,2^{-r\,D^\infty_{\min}(\psi)}\right]^n+1\tag{D10}$$

which grows exponentially in n. This also leads to a general insight about how fast the errors can decay in general distillation protocols, which we formalize as follows.

Corollary A3. If $\Omega_{\mathcal{F}}(\rho)<\infty$ (in particular for every full-rank ρ), then there does not exist any distillation protocol $\rho\to\psi$ such that the error decreases faster than exponentially, even in the probabilistic setting. Specifically, $\varepsilon_n=2^{-O(n)}$ for every physical distillation protocol.

APPENDIX E: ACHIEVABILITY FOR AFFINE THEORIES

Recall that we distinguish two types of resource theories.

(i) *Affine resources*, that is, those for which the set of free states $\mathcal{F}$ contains all states in the affine hull $\mathrm{aff}(\mathcal{F})$ (smallest affine subspace that contains $\mathcal{F}$), i.e., $\mathcal{F}=\mathcal{D}(\mathcal{H})\cap\mathrm{aff}(\mathcal{F})$. Note that every such $\mathcal{F}$ will have an empty interior as a subset of $\mathcal{D}(\mathcal{H})$, since for any set with a nonempty interior $\mathrm{aff}(\mathcal{F})$ would simply be the whole space of Hermitian operators.

(ii) *Full-dimensional resources*, that is, those for which $\mathcal{F}$ has a nonempty interior as a subset of $\mathcal{D}(\mathcal{H})$. Equivalently, these are the resources for which $R_{s,\mathcal{F}}(\rho)<\infty$ for every state.

Proposition 4. Consider any affine resource theory satisfying Axioms I–IV. Then, for all states ρ and ω such that $D^\infty_{\mathcal{F}}(\omega)>0$, the transformation rate under resource-nongenerating operations $\mathcal{O}$ satisfies

$$r_{\mathrm{prob}}(\rho\to\omega)=\frac{\mathbb{D}^\infty_{\Omega,\mathcal{F}}(\rho)}{D^\infty_{\mathcal{F}}(\omega)}.\tag{E1}$$

When $\mathcal{F}$ consists of a single state, then

$$r_{\mathrm{prob}}(\rho\to\omega)=r^\dagger_{\mathrm{prob}}(\rho\to\omega)=\frac{\mathbb{D}_\Omega(\rho\|\sigma)}{D(\omega\|\sigma')}.\tag{E2}$$

Proof. If $\mathbb{D}^\infty_{\Omega,\mathcal{F}}(\rho)=\infty$, then also $\mathbb{D}_{\Omega,\mathcal{F}}(\rho)=\infty$, which means that ρ can be converted into *any* other state probabilistically [30], making the transformation rate unbounded. We can thus assume that $\mathbb{D}^\infty_{\Omega,\mathcal{F}}(\rho)<\infty$ in what follows.

Fix an arbitrary $\delta>0$ and consider the rate

$$r=\frac{\mathbb{D}^\infty_{\Omega,\mathcal{F}}(\rho)}{D^\infty_{\mathcal{F}}(\omega)}\frac{1}{1+\delta}.\tag{E3}$$

Let $(\omega_n)_n$ be a sequence of states such that, for sufficiently small $\varepsilon>0$ and for n large enough, it holds that

$$\tfrac{1}{2}\|\omega_n-\omega^{\otimes\lfloor rn\rfloor}\|_1\leqslant\varepsilon\tag{E4}$$

and

$$\begin{aligned}\frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}_{\lfloor rn\rfloor}}(\omega_n)&\leqslant(1+\delta)\lim_{\varepsilon\to0}\lim_{n\to\infty}\min_{\frac{1}{2}\|\omega'-\omega^{\otimes\lfloor rn\rfloor}\|_1\leqslant\varepsilon}\frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}_{\lfloor rn\rfloor}}(\omega')\\&=(1+\delta)\,rD^\infty_{\mathcal{F}}(\omega)\\&=\mathbb{D}^\infty_{\Omega,\mathcal{F}}(\rho),\end{aligned}\tag{E5}$$

where in the second line we used Lemma 1 and Eq. (C5), and in the third line we used the definition of r.

But since

$$\mathbb{D}^\infty_{\Omega,\mathcal{F}}(\rho)\leqslant\frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}_n}(\rho^{\otimes n})\tag{E6}$$

due to the subadditivity of $\mathbb{D}_{\Omega,\mathcal{F}}$, we have that in fact

$$\mathbb{D}_{\Omega,\mathcal{F}_{\lfloor rn\rfloor}}(\omega_n)\leqslant\mathbb{D}_{\Omega,\mathcal{F}_n}(\rho^{\otimes n})\tag{E7}$$

for all n such that (E5) holds. As $\mathbb{D}_{\Omega,\mathcal{F}}$ is the unique monotone that completely determines the existence of probabilistic transformations in all affine theories (Theorem 5 in [30]), what this entails is that $\rho^{\otimes n}$ can be transformed into ω_n by a probabilistic resource-nongenerating transformation or a sequence thereof; specifically,

$$\forall\zeta>0,\ \exists\Lambda\in\mathcal{O}\ \text{s.t.}\ \frac{1}{2}\left\|\frac{\Lambda(\rho^{\otimes n})}{\mathrm{Tr}\,\Lambda(\rho^{\otimes n})}-\omega_n\right\|_1\leqslant\zeta.\tag{E8}$$

We have thus established the existence of a protocol that, for all sufficiently large n, takes n copies of ρ to a state ω_n

that approximates $\omega^{\otimes \lfloor rn \rfloor}$ arbitrarily closely. Since this holds for every rate r satisfying

$$r < \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D^{\infty}_{\mathcal{F}}(\omega)}, \tag{E9}$$

taking the supremum over all such rates concludes the proof. ∎

APPENDIX F: ACHIEVABILITY FOR NONAFFINE THEORIES

Proposition 5. Consider any resource theory satisfying Axioms I–IV such that $R_{s,\mathcal{F}}(\rho) < \infty$ for all states. Then, for all states ρ and ω such that $\mathbb{D}^{\infty,\bullet}_{s,\mathcal{F}}(\omega) > 0$, the transformation rate under resource-nongenerating operations $\mathcal{O}$ satisfies

$$r_{\text{prob}}(\rho \to \omega) \geqslant \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{\mathbb{D}^{\infty,\bullet}_{s,\mathcal{F}}(\omega)}, \tag{F1}$$

where

$$\mathbb{D}^{\infty,\bullet}_{s,\mathcal{F}}(\omega) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \min_{\frac{1}{2}\|\omega'-\omega^{\otimes n}\|_1 \leqslant \varepsilon} \frac{1}{n} \log[1 + R_{s,\mathcal{F}}(\omega')]. \tag{F2}$$

The approach to proving this result will be analogous to the proof of the affine case (Proposition 4). A key step in that proof was the fact that

$$\mathbb{D}_{\Omega,\mathcal{F}}(\rho) \geqslant \mathbb{D}_{\Omega,\mathcal{F}}(\omega) \Rightarrow \rho \text{ can be converted to } \omega \text{ probabilistically}, \tag{F3}$$

which we used in Eqs. (E7) and (E8). However, this condition is only valid in affine theories. The corresponding condition in nonaffine theories is (Theorem 7 in [30])

$$\mathbb{D}_{\Omega,\mathcal{F}}(\rho) \geqslant \mathbb{D}_{\Omega,s,\mathcal{F}}(\omega) \Rightarrow \rho \text{ can be converted to } \omega \text{ probabilistically}, \tag{F4}$$

where $\mathbb{D}_{\Omega,s,\mathcal{F}}$ is a slightly different variant of the projective relative entropy (based on the "free projective robustness" $\Omega^{\mathcal{F}}_{\mathcal{F}}$ [12]), defined as

$$\begin{aligned}
\mathbb{D}_{\Omega,s,\mathcal{F}}(\omega) &:= \min_{\sigma\in\mathcal{F}} \mathbb{D}_{\Omega,s}(\rho\|\sigma),\\
\mathbb{D}_{\Omega,s}(\rho\|\sigma) &:= \mathbb{D}_s(\rho\|\sigma) + D_{\max}(\sigma\|\rho),\\
\mathbb{D}_s(\rho\|\sigma) &:= \inf\{\lambda \mid \rho \leqslant_{\mathcal{F}} \lambda\sigma\},
\end{aligned} \tag{F5}$$

with $\leqslant_{\mathcal{F}}$ denoting inequality with respect to cone($\mathcal{F}$), i.e., $A \leqslant_{\mathcal{F}} B \iff B - A \in \text{cone}(\mathcal{F})$. The main point to note is that

$$\mathbb{D}_{s,\mathcal{F}}(\rho) := \min_{\sigma\in\mathcal{F}} \mathbb{D}_s(\rho\|\sigma) = \log[1 + R_{s,\mathcal{F}}(\rho)], \tag{F6}$$

which justifies the standard robustness's appearance in Proposition 5.

The proof then proceeds in two steps, which we state as two lemmas for clarity.

Lemma A4. Consider the smoothed regularization of $\mathbb{D}_{\Omega,s,\mathcal{F}}$, namely,

$$\mathbb{D}^{\infty,\bullet}_{\Omega,s,\mathcal{F}}(\omega) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \min_{\frac{1}{2}\|\omega'-\omega^{\otimes n}\|_1 \leqslant \varepsilon} \frac{1}{n} \mathbb{D}_{\Omega,s,\mathcal{F}}(\omega'). \tag{F7}$$

It then holds that

$$\begin{aligned}
\mathbb{D}^{\infty,\bullet}_{\Omega,s,\mathcal{F}}(\omega) &= \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \min_{\frac{1}{2}\|\omega'-\omega^{\otimes n}\|_1 \leqslant \varepsilon} \frac{1}{n} \log[1 + R_{s,\mathcal{F}}(\omega')]\\
&= \mathbb{D}^{\infty,\bullet}_{s,\mathcal{F}}(\omega).
\end{aligned} \tag{F8}$$

Proof. The proof of this statement is completely analogous to that of our Lemma 1, with $D_{\max}$ replaced with $\mathbb{D}_s$; for clarity, let us go through the argument in detail.

We start by observing that since $\mathbb{D}_{\Omega,s}(\rho\|\sigma) \geqslant \mathbb{D}_s(\rho\|\sigma)$ by construction we see that $\mathbb{D}_{\Omega,s,\mathcal{F}}(\omega') \geqslant \mathbb{D}_{s,\mathcal{F}}(\omega') = \log[1 + R_{s,\mathcal{F}}(\omega')]$ and in turn that $\mathbb{D}^{\infty,\bullet}_{\Omega,s,\mathcal{F}}(\omega) \geqslant \mathbb{D}^{\infty,\bullet}_{s,\mathcal{F}}(\omega)$. Therefore, it suffices to show the opposite inequality.

Fix $\delta > 0$. For all sufficiently small $\varepsilon > 0$, one can find a sequence of states $(\omega'_n)_n$ with the property that (a) $\frac{1}{2}\|\omega'_n - \omega^{\otimes n}\|_1 \leqslant \frac{\varepsilon}{2}$ and (b) there exists a sequence $(\sigma_n)_n$ of states $\sigma_n \in \mathcal{F}$ such that

$$\left| \lim_{n\to\infty} \frac{1}{n} \mathbb{D}_s(\omega'_n\|\sigma_n) - \mathbb{D}^{\infty,\bullet}_{s,\mathcal{F}}(\omega) \right| \leqslant \delta. \tag{F9}$$

For $\eta := \log(2\varepsilon^{-1} - 1)$, construct the states

$$\xi_n := \frac{2^{-\eta}\sigma_n + \omega'_n}{1 + 2^{-\eta}}. \tag{F10}$$

On the one hand, it holds that

$$\begin{aligned}
\frac{1}{2}\|\xi_n - \omega'_n\|_1 &= \frac{1}{2}\left\| \frac{2^{-\eta}\sigma_n + \omega'_n}{1 + 2^{-\eta}} - \frac{(1 + 2^{-\eta})\omega'_n}{1 + 2^{-\eta}} \right\|_1\\
&= \frac{1}{2} \frac{2^{-\eta}}{1 + 2^{-\eta}} \|\sigma_n - \omega'_n\|_1\\
&\leqslant \frac{2^{-\eta}}{1 + 2^{-\eta}}\\
&= \frac{\varepsilon}{2},
\end{aligned} \tag{F11}$$

entailing that

$$\tfrac{1}{2}\|\xi_n - \omega^{\otimes n}\|_1 \leqslant \tfrac{1}{2}\|\xi_n - \omega'_n\|_1 + \tfrac{1}{2}\|\omega'_n - \omega^{\otimes n}\|_1 \leqslant \varepsilon. \tag{F12}$$

On the other,

$$\xi_n \leqslant_{\mathcal{F}} \frac{2^{-\eta}\sigma_n + 2^{\mathbb{D}_s(\omega'_n\|\sigma_n)}\sigma_n}{1 + 2^{-\eta}} = \frac{2^{-\eta} + 2^{\mathbb{D}_s(\omega'_n\|\sigma_n)}}{1 + 2^{-\eta}}\sigma_n, \tag{F13}$$

and also

$$\sigma_n \leqslant \sigma_n + 2^{\eta}\omega'_n = (1 + 2^{-\eta})\frac{\sigma_n + 2^{\eta}\omega'_n}{1 + 2^{-\eta}} = (2^{\eta} + 1)\xi_n. \tag{F14}$$

Putting it all together,

$$\begin{aligned}
\frac{1}{n}\mathbb{D}_{\Omega,s,\mathcal{F}}(\xi_n) &\leqslant \frac{1}{n}\mathbb{D}_{\Omega,s}(\xi_n\|\sigma_n)\\
&= \frac{1}{n}[\mathbb{D}_s(\xi_n\|\sigma_n) + D_{\max}(\sigma_n\|\xi_n)]\\
&\leqslant \frac{1}{n}\left(\log \frac{2^{-\eta} + 2^{\mathbb{D}_s(\omega'_n\|\sigma_n)}}{1 + 2^{-\eta}} + \log(2^{\eta} + 1) \right)\\
&= \frac{1}{n} \log(1 + 2^{\eta}\, 2^{\mathbb{D}_s(\omega'_n\|\sigma_n)})\\
&\leqslant \frac{1}{n} \mathbb{D}_s(\omega'_n\|\sigma_n) + \frac{1}{n}\log(1 + 2^{\eta}).
\end{aligned} \tag{F15}$$

Taking the limit then gives

$$\begin{aligned}
&\limsup_{n\to\infty} \min_{\frac{1}{2}\|\omega'-\omega^{\otimes n}\|_1\leqslant\varepsilon} \frac{1}{n}\mathbb{D}_{\Omega,s,\mathcal{F}}(\omega') \\
&\leqslant \limsup_{n\to\infty} \frac{1}{n}\mathbb{D}_{\Omega,s,\mathcal{F}}(\xi_n) \\
&\leqslant \limsup_{n\to\infty}\Big(\frac{1}{n}\,\mathbb{D}_s(\omega_n'\|\sigma_n)+\frac{1}{n}\log(1+2^{\eta})\Big) \\
&= \limsup_{n\to\infty}\frac{1}{n}\,\mathbb{D}_s(\omega_n'\|\sigma_n) \leqslant \mathbb{D}^{\infty,\bullet}_{s,\mathcal{F}}(\omega)+\delta. \qquad (F16)
\end{aligned}$$

Since the above holds for all sufficiently small $\varepsilon > 0$, we see that

$$\begin{aligned}
\mathbb{D}^{\infty,\bullet}_{\Omega,s,\mathcal{F}}(\omega) &= \lim_{\varepsilon\to 0}\limsup_{n\to\infty} \min_{\frac{1}{2}\|\omega'-\omega^{\otimes n}\|_1\leqslant\varepsilon} \frac{1}{n}\mathbb{D}_{\Omega,s,\mathcal{F}}(\omega') \\
&\leqslant \mathbb{D}^{\infty,\bullet}_{s,\mathcal{F}}(\omega)+\delta. \qquad (F17)
\end{aligned}$$

As $\delta > 0$ is arbitrary, this shows that

$$\mathbb{D}^{\infty,\bullet}_{\Omega,s,\mathcal{F}}(\omega) \leqslant \mathbb{D}^{\infty,\bullet}_{s,\mathcal{F}}(\omega), \qquad (F18)$$

completing the proof. ■

Lemma A5. It holds that

$$r_{\text{prob}}(\rho\to\omega) \geqslant \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{\mathbb{D}^{\infty,\bullet}_{\Omega,s,\mathcal{F}}(\omega)}. \qquad (F19)$$

Proof. The proof of this statement is analogous to the proof of Proposition 4 with $\mathbb{D}^{\infty,\bullet}_{\Omega,\mathcal{F}}(\omega)$ replaced with $\mathbb{D}^{\infty,\bullet}_{\Omega,s,\mathcal{F}}(\omega)$. Explicitly, fix $\delta > 0$ and consider any rate

$$r = \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{\mathbb{D}^{\infty,\bullet}_{\Omega,s,\mathcal{F}}(\omega)}\,\frac{1}{1+\delta}. \qquad (F20)$$

Let $(\omega_n)_n$ be a sequence of states such that $\frac{1}{2}\|\omega_n-\omega^{\otimes\lfloor rn\rfloor}\|_1\leqslant\varepsilon$ and such that

$$\begin{aligned}
&\frac{1}{n}\mathbb{D}_{\Omega,s,\mathcal{F}_{\lfloor rn\rfloor}}(\omega_n) \\
&\leqslant (1+\delta)\lim_{\varepsilon\to 0}\lim_{n\to\infty} \min_{\frac{1}{2}\|\omega'-\omega^{\otimes\lfloor rn\rfloor}\|_1\leqslant\varepsilon} \frac{1}{n}\mathbb{D}_{\Omega,s,\mathcal{F}_{\lfloor rn\rfloor}}(\omega') \\
&\leqslant (1+\delta)\,r\,\mathbb{D}^{\infty,\bullet}_{\Omega,s,\mathcal{F}}(\omega) = \mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho), \qquad (F21)
\end{aligned}$$

where the second inequality follows from the weak subadditivity of $\mathbb{D}^{\infty,\bullet}_{\Omega,s,\mathcal{F}}$, specifically the fact that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\limsup_{n\to\infty} \min_{\frac{1}{2}\|\omega'-\omega^{\otimes\lfloor rn\rfloor}\|_1\leqslant\varepsilon} \frac{1}{n}\mathbb{D}_{\Omega,s,\mathcal{F}}(\omega') \\
&= \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{\lfloor rn\rfloor}{n} \min_{\frac{1}{2}\|\omega'-\omega^{\otimes\lfloor rn\rfloor}\|_1\leqslant\varepsilon} \frac{1}{\lfloor rn\rfloor}\mathbb{D}_{\Omega,s,\mathcal{F}}(\omega') \\
&\leqslant r\lim_{\varepsilon\to 0}\limsup_{n\to\infty} \min_{\frac{1}{2}\|\omega'-\omega^{\otimes n}\|_1\leqslant\varepsilon} \frac{1}{n}\mathbb{D}_{\Omega,s,\mathcal{F}}(\omega') \qquad (F22)
\end{aligned}$$

which follows from the definition of lim sup.[4] This implies that

$$\mathbb{D}_{s,\mathcal{F}_{\lfloor rn\rfloor}}(\omega_n) \leqslant \mathbb{D}_{\Omega,s,\mathcal{F}_n}(\rho^{\otimes n}), \qquad (F23)$$

which means that the transformation from $\rho^{\otimes n}$ to ω_n can be realized probabilistically, up to an arbitrarily small error [30]. ■

Lemmas A4 and A5 combined give that

$$r_{\text{prob}}(\rho\to\omega) \geqslant \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{\mathbb{D}^{\infty,\bullet}_{\Omega,s,\mathcal{F}}(\omega)} = \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{\mathbb{D}^{\infty,\bullet}_{s,\mathcal{F}}(\omega)}, \qquad (F24)$$

which is precisely the statement of Proposition 5.

APPENDIX G: ACHIEVABILITY FOR DISTILLATION

We provide an alternative proof of an achievable bound for distillation that, although slightly weaker than Proposition 5, gives more insight into the transformation errors of the protocol. Here we employ the (nonsmoothed) regularized standard robustness:

$$\mathbb{D}^{\infty}_{s,\mathcal{F}}(\rho) = \lim_{n\to\infty}\frac{1}{n}\log[1+R_{s,\mathcal{F}}(\rho^{\otimes n})]. \qquad (G1)$$

Proposition A6. Consider any resource theory in which $R_{s,\mathcal{F}}(\rho) < \infty$ for every state ρ. Then, for every pure target state ψ, it holds that

$$r_{\text{prob}}(\rho\to\psi) \geqslant \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{\mathbb{D}^{\infty}_{s,\mathcal{F}}(\psi)}, \qquad (G2)$$

and the rate is achievable with error $\varepsilon_n = 2^{-\Omega(n)}$.

Proof. By Theorem 12 in [30], for every ε_n such that

$$R_{s,\mathcal{F}_{\lfloor rn\rfloor}}(\psi^{\otimes\lfloor rn\rfloor}) \leqslant \left(\varepsilon_n^{-1}-1\right)^{-1}\Omega_{\mathcal{F}_n}(\rho^{\otimes n}) \qquad (G3)$$

there exists a one-shot probabilistic protocol $\rho^{\otimes n}\to\psi^{\otimes\lfloor rn\rfloor}$ with error (in fidelity) at most ε_n. Any such protocol may consist of a sequence of free operations, but we can simply assume that for every $\delta_n > 0$ there exists a map $\Lambda_n\in\mathcal{O}$ such that $\frac{\Lambda_n(\rho^{\otimes})}{\operatorname{Tr}\Lambda_n(\rho^{\otimes n})} = \tau_n$ for some τ_n with $\langle\psi^{\otimes\lfloor rn\rfloor}|\tau_n|\psi^{\otimes\lfloor rn\rfloor}\rangle \geqslant 1-\varepsilon_n-\delta_n$.

Let us fix some $\eta > 0$ and define $r = \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{\mathbb{D}^{\infty}_{s,\mathcal{F}}(\psi)}-\eta$. Then, choosing

$$\varepsilon_n^{-1} = \frac{\Omega_{\mathcal{F}_n}(\rho^{\otimes n})}{R_{s,\mathcal{F}_{\lfloor rn\rfloor}}(\psi^{\otimes\lfloor rn\rfloor})}+1 \qquad (G4)$$

so that (G3) is satisfied, we ensure that there exists a protocol that takes ρ to ψ at a rate equal to r, with the transformation error for each n given by $\varepsilon_n+\delta_n$. Since the δ_n are arbitrary, it thus remains to show that $\varepsilon_n\to 0$. This is ensured by the fact that

$$\begin{aligned}
\liminf_{n\to\infty}\varepsilon_n^{-1} &\overset{\text{(i)}}{=} \liminf_{n\to\infty}\frac{\Omega_{\mathcal{F}_n}(\rho^{\otimes n})}{R_{s,\mathcal{F}_{\lfloor rn\rfloor}}(\psi^{\otimes\lfloor rn\rfloor})+1}+1 \\
&\overset{\text{(ii)}}{\geqslant} \liminf_{n\to\infty}\frac{\Omega_{\mathcal{F}_n}(\rho^{\otimes n})}{2^{nr\mathbb{D}^{\infty}_{s,\mathcal{F}}(\psi)+n\mu}}+1 \\
&= \liminf_{n\to\infty}\frac{\Omega_{\mathcal{F}_n}(\rho^{\otimes n})}{2^{n\mathbb{D}^{\infty}_{\Omega}(\rho)-n\mathbb{D}^{\infty}_{s,\mathcal{F}}(\psi)\eta+n\mu}}+1 \\
&\overset{\text{(iii)}}{\geqslant} \liminf_{n\to\infty}\frac{\Omega_{\mathcal{F}_n}(\rho^{\otimes n})}{\Omega_{\mathcal{F}_n}(\rho^{\otimes n})\,2^{-n\mathbb{D}^{\infty}_{s,\mathcal{F}}(\psi)\eta+n\mu}}+1 \\
&= \liminf_{n\to\infty} 2^{n[\mathbb{D}^{\infty}_{s,\mathcal{F}}(\psi)\eta-\mu]}+1 \\
&\overset{\text{(iv)}}{=} \infty, \qquad (G5)
\end{aligned}$$

[4]The quantity $\mathbb{D}^{\infty,\bullet}_{\Omega,s,\mathcal{F}}$ can actually be shown to be weakly additive and not merely subadditive—this can be proved as in the derivation of Eq. (41) in [59], but we do not need this fact here.

where in (i) we were free to add the constant term +1 in the denominator as it is irrelevant asymptotically, in (ii) we picked some $\mu > 0$ and took n large enough so that

$$\frac{1}{n}\log[1 + R_{s,\mathcal{F}_{\lfloor rn\rfloor}}(\psi^{\otimes\lfloor rn\rfloor})] \leqslant r\,\mathbb{D}^{\infty}_{s,\mathcal{F}}(\psi) + \mu, \tag{G6}$$

in (iii) we used the fact that $\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho) \leqslant \frac{1}{n}\log\Omega_{\mathcal{F}}(\rho^{\otimes n})$ for every n by the subadditivity of $\mathbb{D}_{\Omega,\mathcal{F}}$, and in (iv) we observed that by picking $\mu < \mathbb{D}^{\infty}_{s,\mathcal{F}}(\psi)\eta$ we can ensure that the term is unbounded. We thus have that the rate $\frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{\mathbb{D}^{\infty}_{s,\mathcal{F}}(\psi)} - \eta$ is achievable for all $\eta > 0$, and taking the supremum over such rates yields the stated result. ■

It then follows immediately from the upper bound of Proposition 3 coupled with the achievability bounds of Propositions 4 and A6 that, when the given theory is affine and $D^{\infty}_{\min,\mathcal{F}}(\psi) = D^{\infty}_{\mathcal{F}}(\psi)$, or when $D^{\infty}_{\min,\mathcal{F}}(\psi) = \mathbb{D}^{\infty}_{s,\mathcal{F}}(\psi)$, then

$$r_{\text{prob}}(\rho \to \psi) = r^{\dagger}_{\text{prob}}(\rho \to \psi) = \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D^{\infty}_{\min,\mathcal{F}}(\psi)}. \tag{G7}$$

An easier to verify condition for the upper and lower bounds to coincide is as follows.

Corollary A7. Consider any state ψ such that we have the following.

(1) ψ maximizes the max-relative entropy measure: for all n, $D_{\max,\mathcal{F}_n}(\psi^{\otimes n})$ is maximal among all states of the same dimension.

(2) $D_{\min,\mathcal{F}_n}(\psi^{\otimes n}) = n\,D_{\min,\mathcal{F}}(\psi)\ \forall n$.

(3) Either (3a) the given resource theory is affine or (3b) the logarithmic standard robustness equals the max-relative entropy for $\psi^{\otimes n}$; specifically,

$$D_{\max,\mathcal{F}_n}(\psi^{\otimes n}) = \log[1 + R_{s,\mathcal{F}_n}(\psi^{\otimes n})]\ \forall n \tag{G8}$$

(e.g., in the resource theory of entanglement).

Then,

$$r_{\text{prob}}(\rho \to \psi) = r^{\dagger}_{\text{prob}}(\rho \to \psi) = \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D_{\min,\mathcal{F}}(\psi)}. \tag{G9}$$

Proof. The reason for introducing condition 1 is perhaps not immediately clear: we do it because any state ψ which maximizes the max-relative entropy among all states of some dimension necessarily satisfies [34]

$$D_{\max,\mathcal{F}}(\psi) = D_{\mathcal{F}}(\psi) = D_{\min,\mathcal{F}}(\psi), \tag{G10}$$

which is helpful in establishing an equality between our upper and lower bounds.

On the one hand, Proposition 3 ensures that

$$r_{\text{prob}}(\rho \to \psi) \leqslant r^{\dagger}_{\text{prob}}(\rho \to \psi) \leqslant \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D^{\infty}_{\min,\mathcal{F}}(\psi)}. \tag{G11}$$

On the other hand, condition 3a gives

$$r_{\text{prob}}(\rho \to \psi) \overset{\text{(i)}}{\geqslant} \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D^{\infty}_{\mathcal{F}}(\psi)} \overset{\text{(ii)}}{\geqslant} \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D^{\infty}_{\max,\mathcal{F}}(\psi)} \overset{\text{(iii)}}{\geqslant} \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D^{\infty}_{\min,\mathcal{F}}(\psi)} \tag{G12}$$

where we used (i) Proposition 4, (ii) the fact that $D(\rho\|\sigma) \leqslant D_{\max}(\rho\|\sigma)$ for all states [26], and (iii) condition 1 and the ensuing Eq. (G10). Similarly, in the case of condition 3b we get

$$r_{\text{prob}}(\rho \to \psi) \overset{\text{(i)}}{\geqslant} \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{\mathbb{D}^{\infty}_{s,\mathcal{F}}(\psi)} \overset{\text{(ii)}}{=} \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D^{\infty}_{\max,\mathcal{F}}(\psi)} \overset{\text{(iii)}}{\geqslant} \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D^{\infty}_{\min,\mathcal{F}}(\psi)} \tag{G13}$$

where we used (i) Proposition 5, (ii) condition 3b, and (iii) Eq. (G10). Invoking condition 2 concludes the proof. ■

We note that a state does not need to be a maximally resourceful one in order to satisfy $D_{\max,\mathcal{F}}(\psi) = D_{\min,\mathcal{F}}(\psi)$ (a counterexample being, e.g., the Clifford magic states discussed in [60] in the resource theory of nonstabilizer quantum computation), but condition 1 is nevertheless a useful assumption to state as it can be satisfied in any given resource theory.

The additivity of the measures (condition 2) and the equality between the standard and generalized robustness measures (condition 3b) may be more difficult to ensure, depending on the given theory. We state them here because they are satisfied for some of the most important states of interest, and notably for the maximally entangled state in entanglement theory [41,61].

APPENDIX H: ON THE PROBABILITY OF SUCCESS IN THE ASYMPTOTIC LIMIT

Recall that

$$r_{\text{prob}>0}(\rho \to \omega) := \sup\left\{ r \,\middle|\, (\Lambda_n)_n \in \mathcal{O},\ \liminf_{n\to\infty} \operatorname{Tr}\Lambda_n(\rho^{\otimes n}) > 0,\ \lim_{n\to\infty} \frac{1}{2}\left\| \frac{\Lambda_n(\rho^{\otimes n})}{\operatorname{Tr}\Lambda_n(\rho^{\otimes n})} - \omega^{\otimes\lfloor rn\rfloor} \right\|_1 = 0 \right\}. \tag{H1}$$

Proposition 6. The rate of any state transformation with a probability of success that does not asymptotically vanish satisfies

$$r_{\text{prob}>0}(\rho \to \omega) \leqslant \frac{D^{\infty}_{\mathcal{F}}(\rho)}{D^{\infty}_{\mathcal{F}}(\omega)}. \tag{H2}$$

Remark Before delving into the proof of the above result, it is instructive to try to use standard techniques [see, e.g., [45] or Eq. (16) in [62]] and see exactly how and why they fail when applied to the probabilistic case. To this end, consider a sequence of transformations $(\Lambda_n)_n \in \mathcal{O}$ with the property that $p_n := \operatorname{Tr}\Lambda_n(\rho^{\otimes n})$ satisfies that $\liminf_{n\to\infty} p_n =: p > 0$, and moreover $\omega_n := \frac{1}{p_n}\Lambda_n(\rho^{\otimes n})$ has the property that $\varepsilon_n := \frac{1}{2}\|\omega_n - \omega^{\otimes\lfloor rn\rfloor}\|_1$ obeys $\lim_{n\to\infty}\varepsilon_n = 0$. Then one can pick an arbitrary sequence of free states $\sigma_n \in \mathcal{F}$, and define the deterministic maps $\Lambda'_n(X) := \Lambda_n(X) + [1 - \operatorname{Tr}\Lambda_n(X)]\sigma_n$. Since $\Lambda'_n \in \mathcal{O}$ is a deterministic free operation for all n, we can try to apply the standard procedure to this new object. Doing so gives

$$\begin{aligned}D_{\mathcal{F}}(\rho^{\otimes n}) &\geqslant D_{\mathcal{F}}(\Lambda_n'(\rho^{\otimes n}))\\ &= D_{\mathcal{F}}(p_n\omega_n + (1-p_n)\sigma_n)\\ &\geqslant p_n D_{\mathcal{F}}(\omega_n) + (1-p_n)D_{\mathcal{F}}(\sigma_n) - h_2(p_n)\\ &= p_n D_{\mathcal{F}}(\omega_n) - h_2(p_n)\\ &\geqslant p_n\left[D_{\mathcal{F}}(\omega^{\otimes\lfloor rn\rfloor}) - \lfloor rn\rfloor\zeta\varepsilon_n - (1+\varepsilon_n)h_2\left(\frac{\varepsilon_n}{1+\varepsilon_n}\right)\right]\\ &\quad - h_2(p_n). \end{aligned} \tag{H3}$$

Here, the first line is by monotonicity of $D_{\mathcal{F}}$, while the third descends from the observation that $D_{\mathcal{F}}$ is "not too convex" (see, e.g., proof of Lemma 1 in [45]), in turn an elementary consequence of the "not-too-concavity" of the von Neumann entropy, seemingly first established by Lanford and Robinson (Theorem 1 in [63]). Finally, the last line of (H3) is the most delicate. It follows from the asymptotic continuity of $D_{\mathcal{F}}$ as established in Lemma 1 in [45], which states that

$$|D_{\mathcal{F}}(\rho) - D_{\mathcal{F}}(\rho')| \leqslant \kappa\varepsilon + (1+\varepsilon)h_2\left(\frac{\varepsilon}{1+\varepsilon}\right) \tag{H4}$$

for every pair of states $\rho, \rho' \in \mathcal{D}(\mathcal{H})$ at trace distance $\varepsilon := \frac{1}{2}\|\rho-\rho'\|_1$, where $\kappa := \sup_{\xi\in\mathcal{D}(\mathcal{H})} D_{\mathcal{F}}(\xi)$, the supremum being over all states ξ on the same space $\mathcal{H}$ as ρ and ρ'. In our case, since we know by Axiom II in Sec. II A that there exists a full-rank free state $\sigma \geqslant \mu\mathbb{1} > 0$ on the same space as ω such that $\sigma^{\otimes m}$ is also free for every m, calling $\mathcal{H}$ the Hilbert space pertaining to ω we see that

$$\kappa = \sup_{\xi\in\mathcal{D}(\mathcal{H}^{\otimes m})} D_{\mathcal{F}}(\xi) \leqslant \sup_{\xi\in\mathcal{D}(\mathcal{H}^{\otimes m})} D(\xi\|\sigma^{\otimes m}) \leqslant -m\log\mu, \tag{H5}$$

which yields the desired inequality if one sets $m = \lfloor rn\rfloor$ and $\zeta := -\log\mu$.

Now, we can divide both sides of (H3) by n and take the limit $n\to\infty$. We obtain that

$$D_{\mathcal{F}}^{\infty}(\rho) \geqslant prD_{\mathcal{F}}^{\infty}(\omega), \tag{H6}$$

which does translate indeed into an upper bound on $r(\rho\to\omega)$, but still features an explicit dependence on p and therefore yields no nontrivial upper bound on $r_{\text{prob}>0}(\rho\to\omega)$.

The fundamental problem with the above technique is in the application of the not-too-concavity of $D_{\mathcal{F}}$, which makes a coefficient p_n appear in front of the relative entropy on the right-hand side. What we do below, instead, is substantially different: instead of leveraging the monotonicity of $D_{\mathcal{F}}$ directly, we look at the projected relative entropy. As we saw, the key properties of this quantity are that, being invariant under rescaling, it can remove the explicit dependence on probabilities, and furthermore it is connected with the standard relative entropy of resource via our asymptotic equipartition property (Lemma 1).

Proof of Proposition 5. Assume that r is an achievable rate for (H2); that is, there exists a sequence of protocols $(\Lambda_n)_n \in \mathcal{O}$ such that $\frac{\Lambda_n(\rho^{\otimes n})}{\operatorname{Tr}\Lambda_n(\rho^{\otimes n})} = \tau_n$ with $\frac{1}{2}\|\tau_n - \omega^{\otimes\lfloor rn\rfloor}\|_1 = \varepsilon_n \to 0$, and furthermore $\liminf_{n\to\infty}\operatorname{Tr}\Lambda_n(\rho^{\otimes n}) =: p > 0$. Define

$$\mathbb{D}_{\Omega,\mathcal{F}}^{\delta}(\rho) := \min_{\frac{1}{2}\|\rho-\rho'\|_1\leqslant\delta} \mathbb{D}_{\Omega,\mathcal{F}}(\rho'), \tag{H7}$$

where the smoothing is over normalized density matrices ρ'. Defining the generalized trace distance as $D_1(X,Y) := \frac{1}{2}\|X-Y\|_1 + \frac{1}{2}|\operatorname{Tr}(X-Y)|$, this quantity satisfies that $D_1(\Lambda(\rho),\Lambda(\rho')) \leqslant D_1(\rho,\rho')$ under the action of any completely positive and trace nonincreasing map Λ (Proposition 3.8 in [64]). We thus have that, for all states ρ and ρ', it holds that

$$\begin{aligned}&\frac{1}{2}\left\|\frac{\Lambda(\rho)}{\operatorname{Tr}\Lambda(\rho)} - \frac{\Lambda(\rho')}{\operatorname{Tr}\Lambda(\rho')}\right\|_1\\ &\leqslant \frac{1}{2}\left\|\frac{\Lambda(\rho)}{\operatorname{Tr}\Lambda(\rho)} - \frac{\Lambda(\rho')}{\operatorname{Tr}\Lambda(\rho)}\right\|_1 + \frac{1}{2}\left\|\frac{\Lambda(\rho')}{\operatorname{Tr}\Lambda(\rho)} - \frac{\Lambda(\rho')}{\operatorname{Tr}\Lambda(\rho')}\right\|_1\\ &= \frac{1}{2}\left\|\frac{\Lambda(\rho)-\Lambda(\rho')}{\operatorname{Tr}\Lambda(\rho)}\right\|_1 + \frac{1}{2}\left\|\frac{\Lambda(\rho')\operatorname{Tr}\Lambda(\rho') - \Lambda(\rho')\operatorname{Tr}\Lambda(\rho)}{\operatorname{Tr}\Lambda(\rho)\operatorname{Tr}\Lambda(\rho')}\right\|_1\\ &= \frac{1}{2}\left\|\frac{\Lambda(\rho)-\Lambda(\rho')}{\operatorname{Tr}\Lambda(\rho)}\right\|_1 + \frac{1}{2}\left|\frac{\operatorname{Tr}\Lambda(\rho') - \operatorname{Tr}\Lambda(\rho)}{\operatorname{Tr}\Lambda(\rho)}\right| \leqslant \frac{\frac{1}{2}\|\rho-\rho'\|_1}{\operatorname{Tr}\Lambda(\rho)}.\end{aligned} \tag{H8}$$

From this, we immediately have that

$$\mathbb{D}_{\Omega,\mathcal{F}}^{\delta}(\rho) \geqslant \mathbb{D}_{\Omega,\mathcal{F}}^{\delta/\operatorname{Tr}\Lambda(\rho)}\left(\frac{\Lambda(\rho)}{\operatorname{Tr}\Lambda(\rho)}\right) \tag{H9}$$

for all free operations $\Lambda \in \mathcal{O}$, using the monotonicity of $\mathbb{D}_{\Omega,\mathcal{F}}$. Now, observe that for any δ such that $0 < \delta < p$ it holds that

$$\limsup_{n\to\infty}\left(\frac{\delta}{\operatorname{Tr}\Lambda_n(\rho^{\otimes n})} + \varepsilon_n\right) = \frac{\delta}{p} < 1. \tag{H10}$$

Therefore, for all sufficiently large integers n we have that $\delta/\mathrm{Tr}\Lambda_n(\rho^{\otimes n}) + \varepsilon_n < 1$. Hence,

$$\begin{aligned}\mathbb{D}^{\delta}_{\Omega,\mathcal{F}}(\rho^{\otimes n}) &\geqslant \mathbb{D}^{\delta/\mathrm{Tr}\Lambda_n(\rho^{\otimes n})}_{\Omega,\mathcal{F}}(\tau_n)\\ &\geqslant \min_{\frac{1}{2}\|\pi_n-\omega^{\otimes\lfloor rn\rfloor}\|_1\leqslant\varepsilon_n} \mathbb{D}^{\delta/\mathrm{Tr}\Lambda_n(\rho^{\otimes n})}_{\Omega,\mathcal{F}}(\pi_n)\\ &\geqslant \mathbb{D}^{\delta/\mathrm{Tr}\Lambda_n(\rho^{\otimes n})+\varepsilon_n}_{\Omega,\mathcal{F}}(\omega^{\otimes\lfloor rn\rfloor}),\end{aligned} \tag{H11}$$

where we used that for any state π_n' such that $\frac{1}{2}\|\pi_n' - \pi_n\|_1 \leqslant \delta/\mathrm{Tr}\Lambda_n(\rho^{\otimes n})$ it holds that

$$\begin{aligned}\frac{1}{2}\|\omega^{\otimes\lfloor rn\rfloor} - \pi_n'\|_1 &\leqslant \frac{1}{2}\|\omega^{\otimes\lfloor rn\rfloor} - \pi_n\|_1 + \frac{1}{2}\|\pi_n - \pi_n'\|_1\\ &\leqslant \varepsilon_n + \frac{\delta}{\mathrm{Tr}\Lambda_n(\rho^{\otimes n})}.\end{aligned} \tag{H12}$$

This means that

$$\begin{aligned}&\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\mathbb{D}^{\delta}_{\Omega,\mathcal{F}}(\rho^{\otimes n})\\ &\geqslant \lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\mathbb{D}^{\delta/\mathrm{Tr}\Lambda_n(\rho^{\otimes n})+\varepsilon_n}_{\Omega,\mathcal{F}}(\omega^{\otimes\lfloor rn\rfloor})\\ &\geqslant \lim_{\delta\to 0} r \liminf_{n\to\infty}\frac{1}{n}\mathbb{D}^{\delta/\mathrm{Tr}\Lambda_n(\rho^{\otimes n})+\varepsilon_n}_{\Omega,\mathcal{F}}(\omega^{\otimes n})\\ &\geqslant r \lim_{\eta\to 0}\liminf_{n\to\infty}\frac{1}{n}\mathbb{D}^{\eta}_{\Omega,\mathcal{F}}(\omega^{\otimes n}),\end{aligned} \tag{H13}$$

where the second line follows analogously to (F22) and the last line follows since $\delta/\mathrm{Tr}\Lambda_n(\rho^{\otimes n}) + \varepsilon_n \to 0$ as $n \to \infty$ and $\delta \to 0$. Using Lemma 1 gives the statement of the proposition. ■

APPENDIX I: ISOTROPIC STATES

We consider a bipartite quantum system with two subsystems of dimension d. Below we will refer to two different sets of quantum states: the set of separable states SEP, and the set of positive partial transpose (PPT) states $\mathrm{PPT} := \{\sigma \in \mathcal{D}(\mathcal{H}) \mid \sigma^{\Gamma} \in \mathcal{D}(\mathcal{H})\}$, with Γ denoting partial transposition over either of the two subsystems. It is well known that the partial transpose of any separable state is positive [65].

Recall that isotropic states are defined as

$$\rho_p = p\Phi_d + (1-p)\frac{\mathbb{1} - \Phi_d}{d^2 - 1} \tag{I1}$$

and that $\rho_p \in \mathrm{SEP} \iff \rho_p \in \mathrm{PPT} \iff p \in [0, \frac{1}{d}]$ [48].

Proposition A8. For every isotropic state ρ_p with local dimension d and all $n \in \mathbb{N}$,

$$\frac{1}{n}\mathbb{D}_{\Omega,\mathrm{SEP}}\big(\rho_p^{\otimes n}\big) = \mathbb{D}_{\Omega,\mathrm{SEP}}(\rho_p) = \begin{cases}\log\frac{p(d-1)}{1-p} & p \geqslant \frac{1}{d},\\ 0 & p \leqslant \frac{1}{d}.\end{cases} \tag{I2}$$

The same is true if SEP is replaced by the set PPT.

Proof. The separable case follows from the faithfulness of Ω_{SEP}, so assume that $p \geqslant 1/d$. The fact that $\Omega_{\mathrm{SEP}}(\rho_p) \leqslant \frac{p(d-1)}{1-p}$ can then be seen from the feasible solution

$$\rho_p \leqslant pd\,\rho_{1/d}, \quad \rho_{1/d} \leqslant \frac{d-1}{(1-p)d}\rho_p. \tag{I3}$$

Submultiplicativity of Ω_{SEP} and the inclusion $\mathrm{SEP} \subseteq \mathrm{PPT}$ then immediately implies that

$$\Omega_{\mathrm{PPT}_n}\big(\rho_p^{\otimes n}\big)^{1/n} \leqslant \Omega_{\mathrm{SEP}_n}\big(\rho_p^{\otimes n}\big)^{1/n} \leqslant \frac{p(d-1)}{1-p}. \tag{I4}$$

For the other inequality, consider the dual form of Ω_{PPT} (and analogously Ω_{SEP}):

$$\Omega_{\mathrm{PPT}}(\rho) = \sup\{\mathrm{Tr}A\rho \mid \mathrm{Tr}B\rho = 1,\ A, B \geqslant 0, \tag{I5}$$

$$\mathrm{Tr}(B-A)\sigma \geqslant 0\ \forall\sigma \in \mathrm{PPT}\}. \tag{I6}$$

Constructing the feasible solutions

$$A = \frac{d-1}{1-p}\Phi_d, \quad B = \frac{\mathbb{1} - \Phi_d}{1-p} \tag{I7}$$

we see that $\mathrm{Tr}A\rho_p = \frac{p(d-1)}{1-p}$ and $\mathrm{Tr}B\rho_p = 1$, so it remains to show that $B - A$ has a non-negative overlap with any PPT (or separable) state. Consider first that

$$\begin{aligned}\max_{\sigma\in\mathrm{PPT}}\mathrm{Tr}A\sigma &= \max_{\sigma\in\mathrm{PPT}}\mathrm{Tr}A^{\Gamma}\sigma^{\Gamma} \leqslant \max_{\rho\in\mathcal{D}(\mathcal{H})}\mathrm{Tr}A^{\Gamma}\rho = \lambda_{\max}(A^{\Gamma})\\ &= \frac{d-1}{1-p}\lambda_{\max}\left(\frac{1}{d}F\right) = \frac{d-1}{d(1-p)},\end{aligned} \tag{I8}$$

where $\lambda_{\max}$ denotes the largest eigenvalue and F is the swap operator, the eigenvalues of which are ± 1. We also see that

$$\begin{aligned}\min_{\sigma\in\mathrm{PPT}}\mathrm{Tr}B\sigma &= \min_{\sigma\in\mathrm{PPT}} B^{\Gamma}\sigma^{\Gamma} \geqslant \min_{\rho\in\mathcal{D}(\mathcal{H})}\mathrm{Tr}B^{\Gamma}\rho\\ &= \frac{1}{1-p}\lambda_{\min}\left(\mathbb{1} - \frac{1}{d}F\right) = \frac{1}{1-p}\left(1 - \frac{1}{d}\right),\end{aligned} \tag{I9}$$

where $\lambda_{\min}$ stands for the smallest eigenvalue. We thus have $\mathrm{Tr}B\sigma \geqslant \mathrm{Tr}A\sigma$ for every PPT state σ, and hence $\Omega_{\mathrm{PPT}}(\rho_p) \geqslant \frac{p(d-1)}{1-p}$ which means that equality must hold. To conclude the n-copy result, it suffices to notice that $A^{\otimes n}$ and $B^{\otimes n}$ are feasible solutions for $\rho_p^{\otimes n}$: the sufficiency of $A^{\otimes n}$ follows from the fact that $(A^{\otimes n})^{\Gamma} = (A^{\Gamma})^{\otimes n}$ and the eigenvalues of $F^{\otimes n}$ are clearly ± 1, and the sufficiency of $B^{\otimes n}$ follows analogously by noting that $\lambda_{\min}(P)$ is a multiplicative quantity for every positive semidefinite P. This implies that

$$\Omega_{\mathrm{SEP}}(\rho_p^{\otimes n}) \geqslant \Omega_{\mathrm{PPT}}\big(\rho_p^{\otimes n}\big) \geqslant \mathrm{Tr}\rho_p^{\otimes n}A^{\otimes n} = \left(\frac{p(d-1)}{1-p}\right)^n, \tag{I10}$$

which together with (I4) means that equality must hold in the above. ■

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